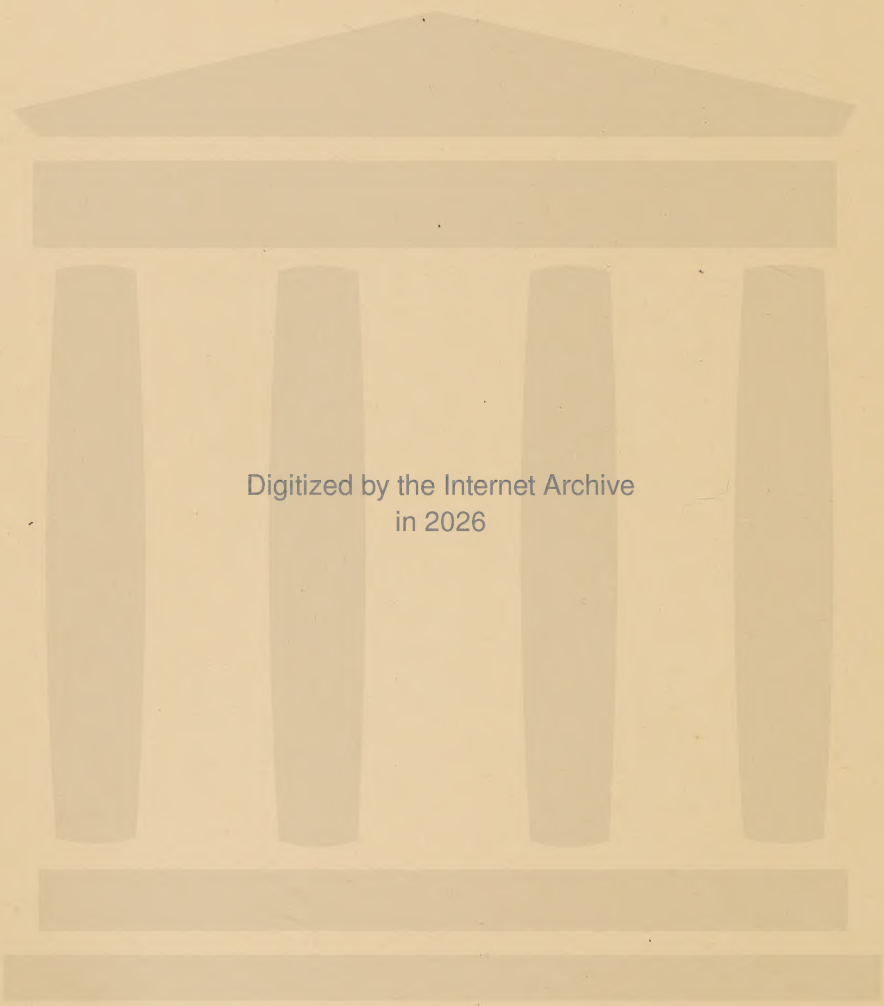


Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1919-1920_18

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 18

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, KINGSWAY, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1920

BIBL.
LONDIN.
UNIV.

CONTENTS.

Records of Proceedings at Meetings, pp. iv–xxx.

Library (Presents and Exchanges), pp. xxxi, xxxii.

Obituary Notices of Bertram Hopkinson (p. xxxiii) and Lord Rayleigh (p. xxxvi).

Papers published in the *Proceedings*, pp. 1–410.

Index of Proceedings, p. 411.

Index of Papers read and Communications made to the Society during the Session 1918–1919, p. 412.

Index of Papers published in this Volume arranged according to the names of Authors, p. 413.

Index of Abstracts of Papers not printed in full in this Volume, arranged according to the names of Authors, p. 414.

Index according to Subjects, p. 415.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1918–JUNE, 1919.

Thursday, November 14th, 1918.

ANNUAL GENERAL MEETING.

Prof. H. M. MACDONALD, President, in the Chair.

Present twenty-four members and three visitors.

Prof. Takeo Wada was elected a member of the Society.

The President announced the deaths of the following members of the Society :—Prof. U. Dini (Hon. For. Member), Prof. O. Henrici (formerly President), Prof. B. Hopkinson, Prof. J. H. Woolston, and Mr. J. W. Sharpe.

The Treasurer presented his Report, which was duly received.

The meeting proceeded to the election of a Council and Officers; the list is as follows :—President, J. E. Campbell; Vice-Presidents, H. Hilton, E. W. Hobson, H. M. Macdonald; Treasurer, A. E. Western; Secretaries, T. J. I'A. Bromwich, G. H. Hardy; other members of the Council, S. Chapman, A. L. Dixon, Miss H. P. Hudson, Sir J. Larmor, J. E. Littlewood, A. E. H. Love, Major P. A. MacMahon, J. W. Nicholson, G. N. Watson.

Prof. H. M. Macdonald, having called Mr. Campbell to the Chair, gave his Presidential Address; it was agreed that the Address should be printed in the *Proceedings*.

Prof. M. J. M. Hill read a paper "On the Use of a Property of Jacobians to Determine the Character of any Solution of an Ordinary Differential Equation of the First Order, or of a Linear Partial Differential Equation of the First Order."

Mr. L. J. Mordell made an informal communication on the expression of a number as the sum of an odd number of squares.

In the absence of the authors, the following papers were communicated by title from the Chair :—

*On the Roots of a certain Equation in Spherical Harmonics : Prof. H. J. Priestley.

*A Detail in Conformal Representation : Mr. J. Hodgkinson.

On the Product of Semiconvergent Series : Mr. T. S. Broderick.

*A Simple Condition for Co-apolar Triangles : Dr. W. P. Milne.

ABSTRACT.

A Statement by Fermat

Mr. L. J. MORDELL.

It is well known that nearly three hundred years ago Fermat, in stating what is now known as Fermat's last theorem, namely that, if n be an integer greater than 2, the equation $x^n+y^n=z^n$ does not admit of integer solutions apart from the trivial case when one of the unknowns is zero, asserted that he had discovered a truly wonderful solution, but that the margin of his book was too small to contain it.

No general proof has been found by mathematicians, even though nine years ago a prize of about £5000 was established for the first proof. Hence the question has been discussed whether Fermat was in possession of a proof. From what is known of his character, it seems certain that, at any rate, he was under the impression that he had a valid proof. It is therefore of interest to analyse other statements made by him.

He had proposed as a problem to the English mathematicians to show that the equation $x^2+2=y^3$ has only one integer solution, evidently given by $x=5$, $y=3$. On this he has a note to the effect that he had discovered an entirely new method—*sane pulcherrima et subtilissima*—which enabled him to solve such questions in integers.†

I have noticed recently that no equation of the form $x^2+a=y^3$ can have more than a finite number of solutions in integers. For, in a paper "Indeterminate Equations of the Third and Fourth Degrees," in the *Quarterly Journal of Mathematics* for 1914, I have shown that all the integer solutions can be found by solving in integers a finite number of

---

* Printed in this volume.

† Cf. Ball's *Mathematical Recreations*, p. 43.

equations of the form $f(x, y) = b$, where $f(x, y)$ is a homogeneous binary quartic in x, y. It follows therefore, as a particular case of a theorem of Thue in *Crelle's Journal* for 1909, that such an equation can have only a finite number of integer solutions [except when $f(x, y)$ is a perfect square]; and hence the same holds for the equation $x^2+a = y^3$.

It seems fairly certain that this result was not known to Fermat, and hence his method must have been such as to give him only a few integer solutions. Perhaps it may have been equivalent to some such process as follows: in $x^2+2 = y^3$ put $y = a^2+2b^3$, and take

$$x+\sqrt{-2} = (a+b\sqrt{-2})^3.$$

By equating imaginary parts,

$$1 = 3a^2b-2b^3 = b\,(3a^2-2b^2),$$

whence $a = 1$, $b = 1$, so that $y = 3$, $x = 5$. In any case, it appears doubtful if Fermat's description of his method would apply at the present time.

Thursday, December 12th, 1918.

Mr. J. E. CAMPBELL, President, in the Chair.

Present eleven members and three visitors.

Mr. H. W. Turnbull, Capt. J. McPolin, and Mr. S. R. U. Savoor, were elected members of the Society.

The Auditor (Lt.-Col. Cunningham) presented his report: on the motion of Mr. Mordell, seconded by Dr. Airey, the Treasurer's report was adopted, and the thanks of the Society were voted to the Treasurer and to the Auditor.

Mr. G. H. Hardy communicated a paper (written in conjunction with Mr. J. E. Littlewood) "Applications of the Method of Farey Dissection in the Analytic Theory of Numbers:—(i) A new solution of Waring's Problem; (ii) Proof that every large number is the sum of at most 33 biquadrates; (iii) The Riemann Hypothesis and the expression of a number as the sum of a stated number of primes". Remarks on the communication were made by Major MacMahon and Mr. Mordell.

Mr. B. M. Wilson communicated a paper (written in conjunction with Mr. N. M. Shah) "Numerical Data connected with Goldbach's Theorem".

Lieut. M. Fréchet made an informal communication "Integrals in Abstract Fields".*

ABSTRACTS.

Applications of the Method of Farey Dissection in the Analytic Theory of Numbers

Mr. G. H. HARDY and Mr. J. E. LITTLEWOOD.

It was first proved by Hilbert (*Math. Annalen*, Vol. 67, 1909, pp. 281–305) that, as was asserted over a hundred years earlier by Waring (*Meditationes Algebraicae*, ed. 3, Cambridge, 1782, pp. 349–350), every number n is the sum of at most $g(k)$ positive k-th powers, $g(k)$ being a function of k alone. In this paper a new proof of this theorem is given, based on entirely different ideas drawn from the theory of analytic functions.

It is also shown that every *large* number [that is to say every number greater than a certain number $n_0(k)$, depending on k alone] is the sum of at most

$$k \cdot 2^{k-1}+1$$

positive k-th powers. In particular every large number is the sum of at most 33 fourth, or 193 sixth, powers.

The method used is also applied to other problems. It is shown, for example, that, if we assume the truth of Riemann's hypothesis concerning the roots of the Zeta-function, and of a generalisation of this hypothesis to the whole body of similar functions which occur in the theory of primes, then every large number is the sum of at most five primes.

In all the applications of the method far more is proved than a bare existence theorem: actual asymptotic formulæ are found for the numbers of representations.

A somewhat fuller account of the nature of these researches (in so far as they are concerned with Waring's problem) will be published shortly in the *Quarterly Journal of Mathematics*. Reference should also be made to two short notes, by Messrs. N. M. Shah and B. M. Wilson, and by Messrs. Hardy and Littlewood, which are about to appear in the *Proceedings of the Cambridge Philosophical Society*. These notes deal with the

---

* The substance of this communication has appeared in the *Bulletin de la Société Mathématique de France*.

application of the method to "Goldbach's Theorem" (that every number is the sum of two primes) and other unproved theorems of the same character. Here no rigorous demonstration is offered, even on the basis of an unproved hypothesis; but the method is shown to lead to very remarkable formulæ, which stand the test of numerical comparison with the facts.

Numerical Data connected with Goldbach's Theorem

Mr. N. M. SHAH and Mr. B. M. WILSON.

Messrs. Hardy and Littlewood have suggested the formula

$$\nu(n) \sim \frac{2An}{(\log n)^2} \frac{p-1}{p-2} \frac{q-1}{q-2} \cdots$$

for the number of representations $\nu(n)$ of a number

$$n = 2^a p^a q^b \ldots \quad (a > 0)$$

as the sum of two primes. Here

$$A = \Pi \left\{ 1 - \frac{1}{(p-1)^2} \right\},$$

p running through all odd primes. The paper, which will be published in the *Proceedings of the Cambridge Philosophical Society*, discusses the correspondence of the formula, and other formulæ suggested by Sylvester, Stäckel, and Brun, with the results of computation.

Thursday, January 16th, 1919.

Mr. J. E. CAMPBELL, President, in the Chair.

Present eight members and two visitors.

Messrs. H. C. Kent, Lloyd Woollard, and T. P. Trivedi, were elected members of the Society.

Mr. L. J. Mordell read a paper on "The Value of the Definite Integral

$$\int_{-\infty}^{+\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\, dt."$$

Dr. Bromwich made an informal communication on "Some Operational Solutions in Conduction of Heat."

In the absence of Prof. M. Fréchet, who had been recalled to his University duties in France, his paper on "The Differentials of Functional Operations," was communicated by title from the chair.

ABSTRACT.

Some Operational Solutions in Conduction of Heat

Dr. T. J. I'A. BROMWICH.

In a paper presented to the Society on March 12th, 1914,* a general theory was developed for obtaining solutions from complex integrals. In particular, in § 4, I obtained Heaviside's interpretation of the operational solution

$$x = \frac{F(p)}{\Delta(p)}(P), \tag{1}$$

where p stands for d/dt, and P is a constant impressed force, applied to a system undisturbed at $t = 0$.

In a discussion of some recent experiments with thermometers carried in aeroplanes, it has been suggested that the surface-conditions are expressed by supposing the surface subjected to a temperature proportional to t.† This leads to equations of the type (1) with $P = Gt$, where G is a constant. To obtain an interpretation, we may write

$$P = Gt = \frac{G}{2\pi\iota}\int \frac{e^{\lambda t}}{\lambda^2}\,d\lambda, \tag{2}$$

where the path encloses $\lambda = 0$; and so (see, for instance, p. 419 of my former paper) equation (1) leads to

$$x = \frac{G}{2\pi\iota}\int \frac{F(\lambda)}{\Delta(\lambda)}\,\frac{e^{\lambda t}}{\lambda^2}\,d\lambda, \tag{3}$$

where the path encloses $\lambda = 0$ and all the roots $\lambda = \alpha$ of $\Delta(\lambda) = 0$.

---

* See *Records of Proceedings at Meetings*, Ser. 2, Vol. 13 (1914), p. xxviii; and "Normal Coordinates in Dynamical Systems," Ser. 2, Vol. 15 (1916), p. 401.

† A. R. McLeod, *Phil. Mag.*, 6th Series, Vol. 37, January 1919, p. 134.

To express (3) in a simple form, suppose that for small values of λ we can write

$$\frac{F(\lambda)}{\Delta(\lambda)} = N_0 + N_1\lambda + N_2\lambda^2 + \dots .$$

Then the residue at $\lambda = 0$ is found to be

$$N_0 t + N_1,$$

and so (3) leads to the result

$$x = \left(N_0 t + N_1 + \Sigma_a \frac{F(a)}{a^2\Delta'(a)} e^{at}\right) G, \tag{4}$$

which corresponds to Heaviside's equation

$$x = \left(N_0 + \Sigma_a \frac{F(a)}{a\Delta'(a)} e^{at}\right) P$$

when P is constant.

As an illustration of the applications of equation (4), let us consider one of the problems discussed by A. R. McLeod in the paper quoted—

A sphere ($r = c$) *is supposed to have its surface maintained at temperature* $u = Gt$.

The differential equation for u can be written

$$\frac{\partial}{\partial t}(ru) = a^2 \frac{\partial^2}{\partial r^2}(ru) ;$$

and if we write

$$q^2 = pc^2/a^2, \tag{5}$$

we obtain the operational solution

$$\frac{ru}{c} = \frac{\sinh(qr/c)}{\sinh q}(Gt). \tag{6}$$

The observations refer to $\bar{u}$, the mean value of u, viz.

$$\bar{u} = \frac{3}{c^3}\int_0^c r^2 u\, dr.$$

By integrating (6) we obtain the symbolic equation

$$\bar{u} = 3\left(\frac{1}{q}\coth q - \frac{1}{q^2}\right)(Gt). \tag{7}$$

We now interpret (7) by means of (4); thus we have, when $|q|$ is small,

$$q\coth q = 1 + \tfrac{1}{3}q^2 - \tfrac{1}{45}q^4 + \dots,$$

so that $$3\left(\frac{1}{q}\coth q-\frac{1}{q^2}\right)=1-\tfrac{1}{15}q^2+\ldots=1-\tfrac{1}{15}\frac{pc^2}{a^2}+\ldots.$$

Hence $$N_0=1,\quad N_1=-\tfrac{1}{15}\frac{c^2}{a^2}. \tag{8}$$

To express (7) in the form used in (4), we can write

$$F(p)=3(q\cosh q-\sinh q)/q^3,$$

$$\Delta(p)=\sinh q/q,$$

because each of these functions is expressible as a series of positive integral powers of p. The roots α of the $\Delta(p)=0$ are given by

$$q=n\pi\iota,\qquad p=-n^2\pi^2a^2/c^2=\alpha, \tag{9}$$

say. For this value of α we get

$$\frac{F(\alpha)}{\Delta'(\alpha)}=\frac{3\cosh q}{q\cosh q\,(dq/dp)}=3\frac{dp}{q\,dq}=\frac{6a^2}{c^2}, \tag{10}$$

by using (5). Thus, on substituting from (8), (9), (10) in equation (4), we obtain the formula for the mean temperature

$$\bar{u}=G\left(t-\frac{c^2}{15a^2}+\Sigma_\alpha\frac{6a^2}{c^2\alpha^2}e^{\alpha t}\right)$$

$$=G\left(t-\frac{c^2}{15a^2}+\frac{6c^2}{\pi^4a^2}\sum_{n=1}^{\infty}\frac{1}{n^4}e^{-n^2\pi^2a^2t/c^2}\right). \tag{11}$$

The formula (11) agrees with that found on p. 136 of A. R. McLeod's paper.

It is easy to apply the same method to the corresponding cylindrical problem; then (6) is replaced by

$$u=\frac{I_0(qr/c)}{I_0(q)}(Gt), \tag{6'}$$

where $$I_0(x)=1+\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}+\ldots$$

is the modified Bessel function of zero order.

Then $$\bar{u}=\frac{2}{c^2}\int_0^c ru\,dr=\frac{2}{q}\frac{I_0'(q)}{I_0(q)}(Gt). \tag{7'}$$

It will be found that, for small values of $|q|$,

$$\frac{2}{q}\frac{I_0'(q)}{I_0(q)} = \frac{1+\frac{1}{8}q^2+\ldots}{1+\frac{1}{4}q^2+\ldots} = 1-\tfrac{1}{8}q^2+\ldots = 1-\tfrac{1}{8}\frac{pc^2}{a^2}+\ldots,$$

so that

$$N_0 = 1, \quad N_1 = -\tfrac{1}{8}\frac{c^2}{a^2}. \tag{8'}$$

Also the roots of the denominator in (7′) are given by

$$q = \iota\omega, \quad p = -\omega^2 a^2/c^2 = \alpha, \tag{9'}$$

where

$$J_0(\omega) = 0.$$

Then

$$\frac{F(\alpha)}{\Delta'(\alpha)} = \frac{2}{q}\frac{I_0'(q)}{I_0'(q)(dq/dp)} = \frac{2dp}{q\,dq} = \frac{4a^2}{c^2}. \tag{10'}$$

And so finally the mean temperature is given by

$$\bar{u} = G\left(t-\frac{c^2}{8a^2}+\frac{4c^2}{a^2}\Sigma\frac{1}{\omega^4}e^{-a^2\omega^2 t/c^2}\right), \tag{11'}$$

agreeing with the formula given on p. 140 of McLeod's paper.

It is easy to obtain corresponding formulæ when account is taken on surface-conductivity; but to save space they will not be investigated here, as the details will appear in a forthcoming number of the *Philosophical Magazine*.

Thursday, February 13th, 1919.

Mr. J. E. CAMPBELL President, in the Chair.

Present seven members.

Miss L. M. Swain was elected a member of the Society.

Mr. G. H. Hardy, as Secretary, reported as to the membership of the Society. During the Session 1917–18, four new members were elected; two members resigned and four members died. At the beginning of the Session 1917–18 the number of members was 296; thus at the beginning of the Session 1918–19 the number was 294.

Mr. T. C. Lewis read a paper "General Pentaspherical Coordinates."

In the absence of the author the following paper was communicated by title from the Chair :—

*Diffraction of Waves by a Wedge of any Angle : Prof. H. S. Carslaw.

ABSTRACT.

General Pentaspherical Coordinates

Mr. T. C. LEWIS.

1. The ratios of the power to radius for a finite point with regard to any five spheres (whose centres are non-cospherical, and no four co-planar) are the pentaspherical coordinates of the point. If the spheres form an orthogonal system, the five centres must be such that each is the orthocentre of a tetrahedron of which the remaining four are the vertices. If the five centres be not so related, it is still known that the sum of the reciprocals of the powers of each point with regard to the circumsphere of the other four is zero. Take as spheres of reference spheres with radii $\rho_1, \ldots, \rho_5$, such that $3\rho_1^2$ is equal to the power of the first point with regard to the circumsphere of the other four, and similarly for ρ_2^2, &c.

Then whether the system be orthogonal or not the following two relations hold, viz.,

$$\Sigma \frac{1}{\rho_r^2} = 0, \tag{1}$$

$$\Sigma \frac{x_r}{\rho_r} = -2. \tag{2}$$

2. A second relation between the coordinates necessarily holds. If c_{rs} is the cosine of the angle of the intersection of the rth and sth spheres, the required relation is

$$\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} & x_1 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} & x_2 \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} & x_3 \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} & x_4 \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & 0 \end{vmatrix} = 0, \tag{3}$$

---

* Printed in this volume.

which reduces to $$\Sigma x_r^2 = 0,$$

for an orthogonal system. It is simply an expression of the fact that the distance between two points vanishes when they coincide. For the square of the distance ρ between two points $(x_1, \ldots)$ and $(x_1', \ldots)$ is given by the equation

$$72V^2\rho_5^2\rho^2+\rho_1^2\rho_2^2\rho_3^2\rho_4^2 \begin{vmatrix} 1 & c_{12} & \ldots & c_{15} & x_1 \\ c_{12} & 1 & \ldots & c_{25} & x_2 \\ \ldots & & \ldots & & \ldots \\ c_{15} & c_{25} & \ldots & 1 & x_5 \\ x_1' & x_2' & \ldots & x_5' & 0 \end{vmatrix} = 0, \tag{4}$$

where V is the volume of the tetrahedron whose vertices are at the first four points. For an orthogonal system this reduces to the known relation

$$2\rho^2+\Sigma x_r x_r' = 0.$$

3. In the more general, as in the orthogonal, system the equation to any sphere is homogeneous of the first degree, viz.,

$$a_1x_1+a_2x_2+\ldots+a_5x_5 = 0;$$

and if $$\Sigma a_r^2+2\Sigma a_r a_s c_{rs} = 1, \tag{5}$$

the radius is given by $$\frac{1}{\rho} = \Sigma \frac{a_r}{\rho_r}. \tag{6}$$

If there be another sphere $$\Sigma a_r' x_r = 0,$$

then c, the cosine of the angle of intersection of the two spheres, is given by

$$c = \Sigma a_r a_r' + \Sigma a_r a_s' c_{rs}, \tag{7}$$

where the last term on the right vanishes for an orthogonal system.

4. The equation to the plane through the first three vertices is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & \frac{1}{\rho_5} \end{vmatrix} = 0, \tag{8}$$

which for an orthogonal system becomes

$$\rho_4 x_4 = \rho_5 x_5.$$

5. The equation to the sphere having as diameter the join of the first two vertices is

$$\rho_1 x_1 + \rho_2 x_2 = \rho_1 \rho_2 c_{12} \Sigma \frac{x_r}{\rho_r}. \tag{9}$$

6. The equation to the sphere circumscribed to the first four vertices is

$$\begin{vmatrix} \rho_1 x_1 & \rho_2 x_2 & \rho_3 x_3 & \rho_4 x_4 & \rho_4 x_5 \\ -\rho_1^2 & \rho_1^2 - 2\rho_1\rho_2 c_{12} & \cdots & \cdots & \rho_1^2 - 2\rho_1\rho_5 c_{15} \\ \rho_2^2 - 2\rho_1\rho_2 c_{12} & -\rho_2^2 & \cdots & \cdots & \rho_2^2 - 2\rho_2\rho_5 c_{25} \\ \rho_3^2 - 2\rho_1\rho_3 c_{13} & \rho_3^2 - 2\rho_2\rho_3 c_{23} & -\rho_3^2 & \cdots & \rho_3^2 - 2\rho_3\rho_5 c_{35} \\ \rho_4^2 - 2\rho_1\rho_4 c_{14} & \rho_4^2 - 2\rho_2\rho_4 c_{24} & \rho_4^2 - 2\rho_3\rho_4 c_{34} & -\rho_4^2 & \rho_4^2 - 2\rho_4\rho_5 c_{45} \end{vmatrix} = 0, \tag{10}$$

or for an orthogonal system

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 + \rho_4 x_4 - 2\rho_5 x_5 = 0.$$

7. In the most general system of pentaspherical coordinates, where the spheres of reference are entirely independent of one another, instead of being members of a particular complex associated with their centres, let the radius ρ_1^2 be k_1 times the square of the corresponding radius in the complex, &c.

Then the linear relation between the five coordinates of a point is

$$\Sigma \frac{k_r x_r}{\rho_r} = 3 - \Sigma k_r = k' : \tag{11}$$

the quadratic relation remains the same as for the complex of spheres associated with the five centres.

Also $$2\rho_1 \Sigma \frac{k_r c_{1r}}{\rho_r} = -(2k_1 + k'),$$

where r has all values except 1. And similarly for other suffixes.

8. By means of this we obtain the equation to determine ρ, the dis-

tance between two points $(x'_1, \ldots)$ and $(x_1, \ldots)$, viz.,

$$18k'^2V_5^2\rho_5^2\rho^2+\rho_1^2\rho_2^2\rho_3^2\rho_4^2\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} & x_1 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} & x_2 \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} & x_3 \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} & x_4 \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 & x_5 \\ x'_1 & x'_2 & x'_3 & x'_4 & x'_5 & 0 \end{vmatrix} = 0. \qquad (12)$$

Moreover in all cases

$$36V^2+\rho_1^2\rho_2^2\rho_3^2\rho_4^2\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & \frac{1}{\rho_1} \\ c_{12} & 1 & c_{23} & c_{24} & \frac{1}{\rho_2} \\ c_{13} & c_{23} & 1 & c_{34} & \frac{1}{\rho_3} \\ c_{14} & c_{24} & c_{34} & 1 & \frac{1}{\rho_4} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & 0 \end{vmatrix} = 0. \qquad (13)$$

Therefore, in the most general case, the distance between two given points is given by

$$\tfrac{1}{2}k'^2\rho_5^2\rho^2\begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & \frac{1}{\rho_1} \\ c_{12} & 1 & c_{23} & c_{24} & \frac{1}{\rho_2} \\ c_{13} & c_{23} & 1 & c_{34} & \frac{1}{\rho_3} \\ c_{14} & c_{24} & c_{34} & 1 & \frac{1}{\rho_4} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} & \frac{1}{\rho_3} & \frac{1}{\rho_4} & 0 \end{vmatrix} = \begin{vmatrix} 1 & c_{12} & c_{13} & c_{14} & c_{15} & x_1 \\ c_{12} & 1 & c_{23} & c_{24} & c_{25} & x_2 \\ c_{13} & c_{23} & 1 & c_{34} & c_{35} & x_3 \\ c_{14} & c_{24} & c_{34} & 1 & c_{45} & x_4 \\ c_{15} & c_{25} & c_{35} & c_{45} & 1 & x_5 \\ x'_1 & x'_2 & x'_3 & x'_4 & x'_5 & 0 \end{vmatrix}. \qquad (14)$$

This may be written

$$\tfrac{1}{2}k'^2\rho_5^2\rho^2 \begin{vmatrix} |c_{rs}| & \dfrac{1}{\rho_s} \\ \dfrac{1}{\rho_r} & 0 \end{vmatrix}_1^4 = \begin{vmatrix} |c_{rs}| & x_s \\ x'_r & 0 \end{vmatrix}_1$$

When $$\Sigma \frac{1}{\rho_r^2} = 0, \qquad k'^2 = 4;$$

and for an orthogonal system of spheres our result reduces to the known form

$$2\rho^2 + \Sigma x_r x'_r = 0.$$

9. In obtaining the results of this paper the following relation between the joins of six points in 3-space has been proved and used. It is an extension of Cayley's relation between the joins of five points. With the usual notation,

$$\begin{vmatrix} 0 & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{12} & 0 & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{13} & d_{23} & 0 & d_{34} & d_{35} & d_{36} \\ d_{14} & d_{24} & d_{34} & 0 & d_{45} & d_{46} \\ d_{15} & d_{25} & d_{35} & d_{45} & 0 & d_{56} \\ d_{16} & d_{26} & d_{36} & d_{46} & d_{56} & 0 \end{vmatrix} = 0.$$

Thursday, March 13th, 1919.

Mr. J. E. CAMPBELL, President, in the Chair.

Present nine members and a visitor.

Messrs. M. A. Giblett and S. Banerji were elected members of the Society.

Mr. J. Hammond read a paper "On the Solution of the Quintic."

Major P. A. MacMahon read a paper "Divisors of Numbers and their Continuations in the Theory of Partitions."

Mr. G. H. Hardy communicated two papers by Mr. S. Ramanujan

(i) "Congruence Properties of Partitions," (ii) "Algebraic Relations between certain Infinite Products."

In the absence of the author, the following paper was communicated by title from the chair:—

On a simple Algebraic Summation of Gauss's Sums: Mr. L. J. Mordell.

ABSTRACTS.

On the Solution of the Quintic

Mr. J. HAMMOND.

I can form a couple of quintics (or rather three such couples, though there is generally no need to consider more than one of them), viz.,

$$Q_1 = (1, 0, c, d_1, e_1, f_1)(x, 1)^5,$$

and

$$Q_2 = (1, 0, c, d_2, e_2, f_2)(x, 1)^5,$$

such that

$$\tfrac{1}{2}(Q_1+Q_2) = (1, 0, c, d, e, f)(x, 1)^5,$$

$$\tfrac{1}{2}(Q_1-Q_2) = \lambda(gx^2+hx+k),$$

the roots of both Q_1 and Q_2 being linear functions of four unequal quantities p, q, r, s, with rational integral coefficients. I can find p, q, r, s in terms of c, g, h, k, without solving any equation higher than a quadratic; thus solving both Q_1 and Q_2 at once. At the same time the other two couples (Q_3, Q_4) and (Q_5, Q_6) are also solved.

Divisors of Numbers and their Continuations in the Theory of Partitions

Major P. A. MACMAHON.

The sum of the divisors of n is

$$a_{n,1} = \Sigma s,$$

where $sm = n$. Further arithmetical functions of n may be defined by

writing

$$a_{n,2} = \Sigma s_1 s_2 \quad (s_1 m_1 + s_2 m_2 = n),$$
$$a_{n,3} = \Sigma s_1 s_2 s_3 \quad (s_1 m_1 + s_2 m_2 + s_3 m_3 = n),$$

and so forth. The introduction of these functions brings the theory of the divisors of numbers into line with the general theory of partitions. The paper contains a study of their properties, and of those of other similar classes of functions.

Congruence Properties of Partitions

Mr. S. RAMANUJAN.

In a paper published recently in the *Proceedings of the Cambridge Philosophical Society*,* I proved a number of arithmetical properties of $p(n)$, the number of unrestricted partitions of n, and in particular that

$$p(5n+4) \equiv 0 \pmod 5,$$

and

$$p(7n+5) \equiv 0 \pmod 7.$$

Alternative proofs of these two theorems were found afterwards by Mr. H. B. C. Darling.†

I have since found another method which enables me to prove all these properties and a variety of others, of which the most striking is

$$p(11n+6) \equiv 0 \pmod{11}.$$

There are also corresponding properties in which the moduli are powers of 5, 7, or 11; thus

$$p(25n+24) \equiv 0 \pmod{25},$$

$$p(49n+19),\ p(49n+33),\ p(49n+40),\ p(49n+47) \equiv 0 \pmod{49},$$

$$p(121n+116) \equiv 0 \pmod{121}.$$

It appears that there are no equally simple properties for any moduli involving primes other than these three.

The function $\tau(n)$ defined by the equation

$$\sum_1^\infty \tau(n)\, x^n = x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24},$$

---

* Vol. 19 (1919), pp. 207–210.

† *Ibid.*, pp. 217, 218.

also possesses very remarkable arithmetical properties. Thus

$$\tau(5n) \equiv 0 \pmod 5,$$

$$\tau(7n),\ \tau(7n+3),\ \tau(7n+5),\ \tau(7n+6) \equiv 0 \pmod 7,$$

while $$\tau(23n+\nu) \equiv 0 \pmod{23},$$

if ν is any one of the numbers

5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22.

Algebraic Relations between certain Infinite Products

Mr. S. RAMANUJAN.

It was proved by Prof. L. J. Rogers* that

$$G(x) = 1 + \frac{1}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \dots$$

$$= \frac{1}{(1-x)(1-x^6)(1-x^{11})\dots} \times \frac{1}{(1-x^4)(1-x^9)(1-x^{14})\dots},$$

and $$H(x) = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \dots$$

$$= \frac{1}{(1-x^2)(1-x^7)(1-x^{12})\dots} \times \frac{1}{(1-x^3)(1-x^8)(1-x^{13})\dots}.$$

Simpler proofs were afterwards found by Prof. Rogers and myself.†

I have now found an algebraic relation between $G(x)$ and $H(x)$, viz.,

$$H(x)\{G(x)\}^{11} - x^2 G(x)\{H(x)\}^{11} = 1 + 11x\{G(x)H(x)\}^6.$$

Another noteworthy formula is

$$H(x)\,G(x^{11}) - x^2 G(x)\,H(x^{11}) = 1.$$

Each of these formulæ is the simplest of a large class.

---

* *Proc. London Math. Soc.*, Ser. 1, Vol. 25 (1894), pp. 318–343.

† *Proc. Camb. Phil. Soc.*, Vol. 19 (1919), pp. 211–216. A short account of the history the theorems is given by Mr. Hardy in a note attached to this paper.

Thursday, April 24th, 1919.

Mr. J. E. CAMPBELL, President, in the Chair.

Present twelve members.

Messrs. J. G. Morfans, T. S. Broderick, W. G. Bickley, A. J. Thompson, and A. C. Banerji were elected members of the Society.

Mr. G. H. Hardy communicated three papers:—

(1) On Lambert's Series, (2) On the Relations between the Convergence of a Series and its Summability by Cesàro's Means: K. Ananda Rau.

A Tauberian Theorem for Lambert's Series: Messrs. G. H. Hardy and J. E. Littlewood.

Dr. Bromwich communicated a paper by Prof. W. H. Young "On a Formula for an Area."*

Col. Cunningham communicated three theorems with reference to primes of the type $10\varpi \pm 1$.

Prof. Hilton made an informal communication on the method of expressing the coordinates of points on a curve in terms of a parameter.

ABSTRACTS.

On Lambert's Series

Mr. K. ANANDA RAU.

The paper contains a proof of a theorem for "Lambert's series", *i.e.* series of the form

$$\Sigma \frac{a_n x^n}{1-x^n},$$

analogous to theorems proved for ordinary power series by Lasker and Pringsheim. The theorem is as follows:—

If

$$\text{(i)} \quad a > 0,$$

$$\text{(ii)} \quad L(u) = (\log u)^{a_1} (\log \log u)^{a_2} \ldots,$$

$$\text{(iii)} \quad \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n} \sim A n^a L(n);$$

then

$$\Sigma \frac{a_n x^n}{1-x^n} \sim \frac{A a \Gamma(a+1)\, \zeta(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

---

* Printed in this volume.

The case in which $a = 0$ is considered separately.

The series Σb_n may be said to be *summable* (L), to sum B, if

$$(1-x)\,\Sigma \frac{nb_n x^n}{1-x^n} \to B,$$

when $x \to 1$. The paper also contains some discussion of the relation between summability (L) and summability (A), that is to say summability by means of Abel's limit $\lim \Sigma b_n x^n$.

On the Relation between the Convergence of a Series and its Summability by Cesàro's Means

K. ANANDA RAU.

This paper is concerned with certain generalisations of theorems due to Hardy (*Proc. London Math. Soc.*, Ser. 2, Vol. 8, 1909, pp. 301–320), and Landau (*Prac Matematyczno-Fizycznych*, Vol. 21, 1910, pp. 97–177).

The main theorem of the paper is the following:—*Suppose that the sequence* (a_n) *of real numbers is such that, as* $n \to \infty$,

$$a_1+a_2+\ldots+a_n = \chi(n)+o(1),$$

where $\chi(n)$ *is a positive increasing function satisfying*

$$\lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ [\chi(n+n\delta)-\chi(n)] = 0. \tag{1}$$

Let Σa_n *be summable* $(C1)$, *and let* $a_n+a_n \geqslant 0$ *for sufficiently large values of* n. *Then* Σa_n *is convergent.*

A Tauberian Theorem for Lambert's Series

Mr. G. H. HARDY and Mr. J. E. LITTLEWOOD.

The principal theorem is: *if* (i) $a_n > -K$, *where* K *is a constant*; (ii) *the series* $\Sigma a_n x^n$ *is convergent for* $|x| < 1$;

$$\text{(iii)}\quad f(x) = \Sigma \frac{a_n x^n}{1-x^n} \sim \frac{A}{1-x},$$

when $x \to 1$ *through positive values: then the series*

$$\Sigma \frac{a_n}{n}$$

converges to the sum A. The theorem does not appear to be capable of an elementary proof: it is necessary to assume one or other of the theorems in the Analytic Theory of Numbers which have only been proved by the aid of Cauchy's Theorem.

The logical relations of some of these theorems are discussed further. Incidentally it is shown that the existence of the limit

$$\lim_{x \to 1} (1-x) \Sigma \frac{a_n x^n}{1-x^n}$$

necessarily involves that of $\lim_{x \to 1} \Sigma \frac{a_n}{n} x^n$.

This may be expressed, in the terminology used by Ananda Rau, by saying that *a series summable by Lambert's method is in all cases summable by Abel's method.*

On a Formula for an Area

Prof. W. H. YOUNG.

Although considerable progress has been made in recent years in the theory of the area of plane curves, the work done suffers from two defects: (1) the fact has been entirely overlooked, that *area* must, like *length*, be regarded as possessing a sign, and that, in particular, even when the word *area* is used in the ordinary sense, as applied to a portion of the plane enclosed by a simple Jordan curve, it may be convenient to determine it as the algebraic sum, or the limit of the algebraic sum, of a number of positive and negative areas; (2) no satisfactory discussion has been given of the expression of an area as a double integral, even when the area is enclosed by a simple Jordan curve, and traced out by a doubly infinite series of simple Jordan curves, in such a manner as to set up a (1, 1)-correspondence $x = x(u, v)$, $y = y(u, v)$ between the area in the (x, y)-plane and the portion of the auxiliary (u, v)-plane of which it is the image.

In the present paper, the author supplies what is lacking in both these respects. The results obtained are of fundamental importance alike in the theory of the area of surfaces, including the generalisations to which

this theory gives rise, and that of the transformation of the variables in a multiple integral. The author has, in fact, been enabled to base on them an entirely new theory of the area of surfaces, which appears to be free from the very serious objections that have been made to earlier theories. Moreover, the main result of the paper, when interpreted in the light of a general theorem, proved by the author elsewhere, leads to conditions for the validity of the transformation of the variables in a multiple integral, of such generality as to render it probable that no appreciably wider conditions are obtainable.

A definition is first given of the area of a polygonal figure, which may cross itself any number of times. This area F is defined to be half the resultant couple of a system of forces, represented in magnitude, line of action and sense, by the sides of the figure in order. The area A of a closed curve, $x = x(u)$, $y = y(u)$, is then defined as the unique limit when this exists of the area F of a polygonal figure inscribed in it, the vertices of the figure following one another in the order determined by the increase of the parameter u, as the sides of the figure decrease indefinitely in length.

With this definition it is shown that the formula

$$A = \int_a^c \int_b^d \frac{\partial (x, y)}{\partial (u, v)} \, du dv,$$

for the area of the image of the rectangle $(a, c; b, d)$ *given by the correspondence* $x = x(u, v)$, $y = y(u, v)$, *is valid with the single restriction that the partial derivates of* x *and* y *with respect to* u *and* v *are bounded. More generally it is found to be sufficient that*

(i) $x(u, v)$ *and* $y(u, v)$ *should be absolutely convergent integrals with respect to each independent variable separately*;

(ii) $\left|\frac{\partial x}{\partial v}\right| \leqslant \mu(v)$, $\left|\frac{\partial y}{\partial v}\right| \leqslant M(v)$, *except at most for a set of values of* v *of zero content, where* $\mu(v)$ *and* $M(v)$ *are summable functions of* v;

(iii) *the total variations of* $x(u, v)$ *and* $y(u, v)$ *with respect to* u *should be bounded functions of* v, *or, more generally, possess absolutely convergent integrals with respect to* $\int M(v) dv$ *and* $\int \mu(v) dv$ *respectively, in other words that* $\iint \mu(v) \left|\frac{\partial y}{\partial u}\right| du dv$, $\iint M(v) \left|\frac{dx}{dv}\right| du dv$ *should exist.*

Expression of Primes $p = 10\varpi \pm 1$ *in various Forms*

Lt.-Col. ALLAN CUNNINGHAM, R.E.

(i) Let $C = (x^3-y^3) \div (x-y)$, $C' = (x^3+y^3) \div (x+y)$.

Then $2C-C' = x^2+3xy+y^2$, $2C'-C = x^2-3xy+y^2$.

Also $x^2 \pm 3xy+y^2 = (x \pm \frac{3}{2}y)^2 - 5\left(\frac{y}{2}\right)^2 = t^2-5u^2$ (suppose).

Now every $p = 10\varpi \pm 1$ is expressible in the form t^2-5u^2. Hence every such prime can be expressed in the form $2C-C'$ in one way, and in the form $2C'-C$ in an infinite number of ways.

(ii) Let $Q = (x^5-y^5) \div (x-y)$, $Q' = (x^5+y^5) \div (x+y)$,

$$S = (x^6+y^6) \div (x^2+y^2).$$

Then Q, Q', S are known to be expressible in the forms

$$Q = T_q^2-5xyU_q^2, \quad Q' = T_q'^2+5xyU_q'^2,$$

$$S = T_s^2-6xyU_s^2 = T_t'^2+6xyU_s^2,$$

wherein $T_q = x^2+3xy+y^2 = T_s$, $U_q = x+y = U_s$,

$T_q' = x^2-3xy+y^2 = T_s'$, $U' = x-y = U_s'$.

Hence $6Q-5S = (x^2+3xy+y^2)^2$, $6Q'-5S = (x^2-3xy+y^2)^2$.

Hence, by (i), the square of every prime $p = 10\varpi \pm 1$ can be expressed in the form $6Q-5S$ in one way, and in the form $6Q'-5S$ in an infinite number of ways.

(iii) In the above theorems (i), (ii) the symbol p may be replaced by P, where P is any of p, p^κ, $\Pi(p)$, $\Pi(p^\kappa)$, or any of these multiplied by 5 (not 25); or (more shortly) any number of the form t^2-5u^2.

But, when P contains r different primes p, there will be 2^{r-1} different forms $2C-C'$ and $6Q-5S$, and 2^{r-1} different series of forms $2C'-C$ and $6Q'-5S$, corresponding to the 2^{r-1} different series of forms t^2-5u^2.

(iv) Given a number $N = P$ or P^2, it may be expressed in the forms of (i) and (ii) by first expressing it in the form t^2-5u^2. The transformation formula giving x, y in terms of t, u are as follows.

Let $N_1 = x_1^2+3x_1y_1+y_1^2, \quad N_2 = x_2^2-3x_2y_2+y_2^2,$

$$N_3 = 3x_3y_3-x_3^2-y_3^2,$$

and let ω mean an *odd* number, and ϵ mean an *even* number.

The cases of $t > 3u$ and $t < 3u$ require separate formulæ :—

(1) $t > 3u$.

$$x_1 = t-3u, \quad y_1 = 2u, \quad [x_1 = \omega,\ y_1 = \epsilon],$$
$$x_2 = t+3u, \quad y_2 = 2u, \quad [x_2 = \omega,\ y_2 = \epsilon],$$
$$x_3 = t+u, \quad y_3 = t-u, \quad [x_3y_3 = \omega].$$

(2) $t < 3u$.

$$x_1 = 3t-7u, \quad y_1 = 3u-t, \quad [x_1y_1 = \omega],$$
$$x_2 = 2u, \quad y_2 = 3u-t, \quad [x_2 = \epsilon,\ y_2 = \omega],$$
$$x_3 = t-u, \quad y_3 = 2t-4u, \quad [x_3 = \omega,\ y_3 = \epsilon].$$

It is known that, when t, u are minima, then $t > \frac{5}{2}$; this gives all the x, y positive integers. But, if $3t < 7u$, x_1 becomes negative; x_2, x_3 are always positive.

(v) The following table gives the values of t, u (minima), and the corresponding values of (x_1, y_1), (x_2, y_2), (x_3, y_3) for all primes

$$p = t^2-5u^2 < 100 :$$

from these the values of C, C', Q, Q', S are easily computed.

p	t, u	x_1, y_1	x_2, y_2	x_3, y_3	p	t, u	x_1, y_1	x_2, y_2	x_3, y_3
1	1, 0	1, 0	1, 0	1, 1	19	8, 3	3, 1	6, 1	5, 4
11	4, 1	1, 2	7, 2	5, 3	29	7, 2	1, 4	13, 4	9, 5
31	6, 1	3, 2	9, 2	7, 5	59	8, 1	5, 2	11, 2	9, 7
41	11, 4	5, 1	8, 1	7, 6	79	18, 7	5, 3	14, 3	11, 8
61	9, 2	3, 4	15, 4	11, 7	89	13, 4	1, 8	25, 8	17, 9
71	14, 5	7, 1	10, 1	9, 8	5	5, 2	2, 1	4, 1	3, 2
101	11, 2	5, 4	17, 4	13, 9					

Thursday, May 15th, 1919.

Mr. J. E. CAMPBELL, President, in the Chair.

Present twelve members.

Messrs. A. H. S. Gillson, J. Hodgkinson, C. T. Preece, and J. R. Wilton, Miss M. Pick and Miss D. Wrinch were nominated for election.

Dr. G. N. Watson read a paper "The Zeros of Lommel's Polynomials." Dr. Bromwich and Prof. Hobson took part in the discussion of the paper.

A paper by Prof. W. H. Young, "On the Triangulation Method of Defining the Area of a Surface," was communicated by title from the chair.

ABSTRACTS.

The Zeros of Lommel's Polynomials

Dr. G. N. WATSON.

The zeros of Bessel functions of negative order have been investigated by Hurwitz (*Math. Ann.*, Vol. 33), who treated the functions as limiting forms of Lommel's polynomials. The object of the present communication is to point out and rectify some slight defects in the analysis given by Hurwitz. The results obtained by him are so important in the theory of Bessel functions that the proof of them should be made as rigorous as possible.

Thursday, June 12th, 1919.

Mr. J. E. CAMPBELL, President, in the Chair.

Present thirteen members.

Messrs. A. H. S. Gillson, J. Hodgkinson, C. T. Preece, and J. R. Wilton, Miss M. Pick and Miss D. Wrinch were elected members of the Society.

Messrs. C. Chaffer, C. G. Darwin, H. Freeman, W. V. Lovell, and T. Knox Shaw were nominated for election.

Mr. Lloyd Woollard was admitted into the Society.

Mr. L. J. Mordell read a paper "On some Series whose n-th Term involves the Number of Classes of Binary Quadratics of Determinant $-n$."

The following papers were communicated by title from the chair :—

Groups involving Three and only Three Operators which are Square : Prof. G. A. Miller.

*The significance of Apolar Triangles in Elliptic Function Theory : Dr. W. P. Milne and Dr. D. G. Taylor.

On the General Theory of Ruled Surfaces : C. V. Hanumanta Rao.

*The Vibrations and Stability of a Rotating Cylinder : Mr. F. B. Pidduck.

ABSTRACTS.

Groups involving Three and only Three Operators which are Square

Prof. G. A. MILLER.

In 1854 A. Cayley inaugurated abstract definitions of groups by defining abstractly the groups of order 8. About two years later W. R. Hamilton gave abstract definitions of the groups which are simply isomorphic with the groups of movements of the five ancient regular solids. Since this time various other categories of groups have been defined abstractly, including those non-Abelian groups which R. Dedekind defined, in 1897, by the property that every sub-group is invariant, and known as Hamiltonian groups. In the present paper the author defines an additional category of groups by the property that three and only three of their operators are squares of operators of the group. He proves that there are eight infinite systems of such groups. Seven of these systems are similar to the Hamiltonian groups as regards the property that there is one and only one smallest non-Abelian group in the system, and all the other groups of the system are direct products having for one factor this smallest group, while the other factor is an Abelian group. The remaining system is composed of Abelian groups.

* Printed in this volume

The significance of Apolar Triangles in Elliptic Function Theory

Prof. W. P. Milne, M.A., D.Sc., and D. G. Taylor, M.A., D.Sc.

If S be a cubic curve and H its Hessian, the condition that the three points on S whose elliptic parameters are α, β, γ, shall constitute a triad apolar to the curve $\lambda S+H=0$ is

$$\lambda S_{\alpha\beta\gamma}+H_{\alpha\beta\gamma}=0,$$

where
$$S_{\alpha\beta\gamma}\equiv 12\wp(\alpha)\wp(\beta)\wp(\gamma)-\Sigma\wp'(\beta)\wp'(\gamma)-g_2\Sigma\wp(\alpha)-3g_3,$$
$$H_{\alpha\beta\gamma}\equiv \Sigma\wp(\alpha)\wp'(\beta)\wp'(\gamma)-g_2\Sigma\wp(\beta)\wp(\gamma)-3g_3\Sigma\wp(\alpha)-\tfrac{1}{4}g_2^2.$$

It is found that the expression $\lambda S_{\alpha\beta\gamma}+H_{\alpha\beta\gamma}$ can be resolved into two factors Δ, A, where Δ may be called the "collinearity-factor" and A the "apolarity-factor". A is a function of the differences of the parameters $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, and is therefore unaltered if for α, β, γ we substitute $-(\alpha+\tau)$, $-(\beta+\tau)$, $-(\gamma+\tau)$. On the other hand, Δ is not independent of τ if we make the above substitutions. We therefore have the following geometrical result:—

If three points, which are not collinear, lie on a cubic curve and are apolar to a given member of the syzygetic pencil of cubics through the flexes, then the three points obtained by projecting the three given points through a point of the cubic curve on to the curve again are still apolar to the same member of the syzygetic pencil.

Some considerations on the General Theory of Ruled Surfaces

C. V. Hanumanta Rao.

The structure of a ruled surface situated in ordinary space is not at all simple. On the other hand, the structure of a scroll situated in its normal space is extremely simple; and the present paper is devoted to the study of such scrolls. The normal space of a scroll with two directrices is determined in terms of the characteristic numbers of its plane sections. This question is precisely the same as the question of the freedom of the complete system of curves determined by the system of plane sections, and leads to a discussion of the deficiency in freedom of the characteristic series.

The Vibrations and Stability of a Rotating Cylinder

Mr. F. B. PIDDUCK.

In pursuance of a note inserted in *Proceedings*, Ser. 2, Vol. 16 (1918), p. lviii, a revision is made of a paper in *Proceedings*, Ser. 2, Vol. 9 (1910), p. 352, involving serious errors. Southwell's method of investigating elastic stability, by no means restricted to statical problems, is here used to find the period equation to the order of approximation usual in the theory of thin rods. If Ω is the whirling speed for an assigned wave-length and ω the angular velocity, the frequency of vibration about the state of steady motion for the same wave-length is given by the simple equation $p = \omega \pm \Omega$, and the character of the vibration is also simple, including whirling as a special case.

LIBRARY

Presents.

BETWEEN December 31st, 1918, and December 31st, 1919, the following presents were made to the Library by their respective authors and publishers:—

Araujo, Roberto.—Las Curvas *W*.

Castelnuovo, G.—Calcolo delle Probabilità.

Kenyon, T. G.—Education.

Le Nain, L.—Rapport succinct sur l'état du Palais des Académies après le départ des Allemands.

Opperman, A.—Premiers Eléments d'une Théorie du Quadrilatère complet.

Petrovitch, Michel.—Les spectres numériques.

Ventosa, Vincente. — Reflexions acerca de la resolución de las ecuaciones algébricas numéricas por el método de Graffe.

Zeuthen, H. G.—Sur l'origine d'Algèbre. Sur les définitions d'Euclide. Erklärung eines "Paradox" in der abzählenden Geometrie.

Amsterdam: Société Mathématique, Revue semestrielle des Publications Mathématiques, tome 26, 2me partie; tome 27, 1re et 2me parties; tomes 21–25.

Athenæum: nos. 2647–4678.

Journal des Mathématiques pures et appliqués, 7me série, tome 1, année 15, fasc. 4, 1915, also June 1844.

Kyoto: Imperial University, College of Science Memoirs, vol. 3, nos. 5–10.

London: Conjoint Board of Scientific Societies, Confirmed Minutes of Tenth, Eleventh, and Twelfth Meetings; Bulletin of Scientific and Technical Societies; Second Report of Water-Power Committee.

Madrid: Junta para Ampliación de Estudios e Investigaciones Científicas, Trabajos de Investigación y Ampliación de Estudios Organizados para el Curso de 1918, 1919; Publicaciones del Laboratorio y Seminario Matemático, tome 3, memoria 2, 3.

Mathematical Gazette, vol. 4, no. 69, pt. 2; vol. 5, no. 78; vol. 7, no. 111; vol. 8, nos. 123–126; vol. 9, nos. 127–142.

Mathematical Questions and Solutions, vol. 6, pt. 5.

Nouvelles Annales de Mathématiques, 4me série, tome 11, June 1911.

Paris: Ecole Normale Supérieure, Annales Scientifiques, no. 3.

Paris: L'enseignement mathématique, 20me année, nos. 3–5.

Sendai: Tôhuko Imperial University, Science Reports, vol. 7, no. 3; vol. 8, nos. 1, 2.

Sendai: Tôhuko Mathematical Journal, vol. 14, nos. 3, 4; vol. 15; vol. 16, nos. 1, 2.

Tokyo: Imperial University, College of Science, Calendar 1917, 1918; Journal, vol. 41, article 4.

Tokyo: Physico-Mathematical Society of Japan, Proceedings, vol. 9, no. 22; 3rd series, vol. 1, nos. 1–9.

Exchanges.

Acta Mathematica: vol. 42, nos. 1, 2.

American Journal of Mathematics, vol. 40, no. 4; vol. 41, nos. 3, 4.

American Mathematical Society, Transactions, vol. 20, nos. 1–3; Bulletin, vol. 25, nos. 3–10; vol. 26, nos. 1–3.

American Philosophical Society, Proceedings, vol. 57, nos. 6, 7.

Amsterdam: Nieuw Archief voor Wiskunde, deel 12, stuk 3; deel 13, stuk 1.
Amsterdam: Wiskundige Opgaven, deel 12, stuk 6; deel 13, stuk 1.
Berlin: Mathematische Zeitschrift, band 5, hefte 1, 2.
Bologna: Reale Accademia delle Scienze, Memorie, série 6, tomo 10; serie 7, tomo 1–3; Rendiconti, vols. 17–20.
Bordeaux: Société des Sciences Physiques et Naturelles, Mémoires, série 6, tome 5, 2me cahier; série 7, tome 1, 1re cahier; Procès-Verbaux des Séances, 1913–1918.
Calcutta: Indian Association for the Advancement of Science, Bulletin, no. 15; Proceedings, vol. 4, pts. 1, 2; Report and Proceedings of the Science Convention for the year 1917.
Calcutta: Mathematical Society, Proceedings, vol. 19, pt. 5.
Cambridge Philosophical Society, Proceedings, vol. 19, pt. 5.
Cambridge, Mass.: Annals of Mathematics, vol. 4, nos. 2, 3; vol. 12, no. 4; vol. 16, no. 3.
Cape Town: South African Association for the Advancement of Science, Journal, vol. 15, nos. 2–8; Circular, no. 3.
Catania: Accademia Gioenia, Atti, vol. 11; Bolletino, fasc. 45, 46.
Dublin: Royal Irish Academy, Proceedings, vol. 34.
Edinburgh Mathematical Society, vol. 36, pts. 1, 2; vol. 39, pt. 2.
Edinburgh: Royal Society, Proceedings, vol. 36, pts. 3, 4; vol. 38, pts. 1, 2.
Firenze: Biblioteca Nazionale Centrale di Firenze, Bollettino, nos. 209–222.
Institute of Actuaries, Journal, vol. 51, pts. 2, 4; List of Fellows and Additions to Library, 1918, 1919.
Institute of Naval Architects: Fifteen Papers read at the Spring Meetings, 1919.
København: Kgl. Danske Vidensk. Selskab. Mathematisk-fysiske Meddelelser, vol. 1, nos. 9–12, Oversigt, June 1918, May 1919.
La Plata: Universidad Nacional de la Plata, Annuario 1918, 1919; Memoria, 1917, no. 7; Contribución al Estudio de las Ciencias físicas y matemáticos, nos. 35, 36, 38, 41, 42.
London: Nature, vol. 102, nos. 2566–2574; vol. 103; vol. 104, nos. 2601–2617.
London: Physical Society, Proceedings, vol. 31, pts. 2–5.
London: Royal Astronomical Society, Monthly Notices, vol. 79.
London: Royal Society, Philosophical Transactions, vols. 218, 220, nos. 571–575; Proceedings, vol. 95, nos. 668–673; vol. 96, nos. 674–678.
Madras: Indian Mathematical Society, Journal, vol. 10, no. 6; vol. 11, nos. 1–5; and List of Members.
Manchester: Literary and Philosophical Society, Memoirs and Proceedings, vol. 62, pts. 1–3.
Marseille: Faculté des Sciences, Annales, tomes 22, 23; tome 24, fasc. 1, 2.
Oporto: Academia Polytechnica, Annaes Scientificos, vol. 10, no. 4; vol. 11, no. 12.
Palermo: Circolo Matematico, Rendiconti, tomo 42, fasc. 2, 3; tomo 43, fasc. 1.
Paris: Bulletin des sciences mathématiques, tome 42, nos. 9–12; tome 43, nos. 1–9.
Paris: Société Mathématique de France, Bulletin, tome 44, fasc. 2, 3; tome 45, fasc. 2, 3; tome 46, fasc. 2, 3; Comptes Rendus des Séances, année 1919.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 27, fasc. 9–12, 2me semestre; vol. 28, fasc. 1–9, 2me semestre.
Torino: Reale Accademia delle Scienze, Atti, vols. 53, 54, nos. 1–15.
Toronto: Royal Canadian Institute, Transactions, vol. 12, pt. 1.
Toulouse: Annales de la Faculté des Sciences de l'Université, tome 7.
Washington: National Academy of Sciences, Proceedings, vol. 4, no. 12; vol. 5, nos. 1–9.
Washington: U.S. Naval Observatory, Annual Report, 1918.
Zurich: Naturforschende Gesellschaft, Vierteljahrschrift, 61 jahrgang, hefte 1, 2; 62 jahrgang, hefte 1–4; 63 jahrgang, hefte 1, 2; 64 jahrgang, hefte 1, 2.

Purchased.

Messenger of Mathematics, vol. 48, nos. 3–10.
Philosophical Magazine, vols. 37, 38.
Quarterly Journal of Mathematics, vol. 48, no. 3.

OBITUARY NOTICES

PROFESSOR B. HOPKINSON.*

Bertram Hopkinson came up to Trinity College from St. Paul's School in 1892, having been elected to a major scholarship at a very early age. His reputation had preceded him: and much was expected of him on the ground of heredity. For he came of a family famous in Cambridge annals. His father John Hopkinson had been Senior Wrangler in 1871: he then plunged immediately into the career in practical engineering which was suited to the combination of scientific and practical interests for which he was remarkable. His opportunity arrived when the transformation of Mechanical Engineering to an electrical basis rather suddenly became imminent: in the early stages of that great development he and Lord Kelvin above all others were the chief intellectual guiding forces: and the guidance of the evolution of the modern dynamo and electric motor was Dr. Hopkinson's main share in the advance. Under such conditions it was a great advantage to his son to reside chiefly at home, and to have the run of the private laboratory in which fundamental investigations in Magnetism were going on as a sort of by-play to his father's strenuous engineering work. Later when a fully equipped laboratory at King's College, London, was at the disposal of his father as honorary professor, the son's name had begun to appear as joint author before he came up to the University. The relations with his father are brought out with unconscious grace in the biographical memoir prefixed to the edition, prepared by him in two volumes, of his father's *Scientific Papers*.

With such antecedents it was natural that his first aim should be the Mathematical Tripos. Those who had to do with him in that connexion may remember a certain modest and deferential robustness and independence: he had obviously the gift of friendship with his peers: and it could not be said that he sank the manifold interests of University life in one intellectual ambition. He rowed in the Trinity boat at Henley Illness compelled him to pass out with an *aegrotat* degree: but in the follow-

* Reprinted by permission from the *Cambridge Review*, Oct. 1918.

ing year (1896) full amends came in a position in the first division of the first class in the advanced mathematical subjects.

In 1898 he contributed to the *Proceedings of the London Mathematical Society** an academic dissertation "On Discontinuous Fluid Motion involving Sources and Vortices," which will form pp. 1–21 of his *Scientific Papers*.

But with the practical instincts of his family he required a wider field of activity than could come to a junior resident at the University. He accordingly began to prepare for the Bar, doubtless with a view to a scientific practice in questions of patents, for which his father's connexions formed the best possible introduction. The experienced leader in whose chambers he worked has described him as the most brilliant pupil he ever had. An eminent judge with whom he acted recently as scientific assessor warmly appreciated his services and prophesied for him a brilliant future.

He was pursuing a legal career with rising prospects when the loss of his father and a younger brother and two sisters in an Alpine accident in 1898 changed the course of his life. His father's practice, of most interesting and important type, as a consulting engineer, lay vacant: and with his usual unflinching courage the son resolved to abandon the Bar and take it up. With the aid in partnership of an uncle and one of his father's former assistants he had carried it on with success for a few years when a new opening arose.

The chance came to him in 1903 to return to Cambridge in a responsible position. On the departure of Sir Alfred Ewing to take up the office of Director of Education in the Royal Navy, the problem of filling the vacancy in the chair of Engineering was more difficult even than is usual in such cases, where a man of scientific eminence has to be enticed from the activities of a lucrative profession. The electors decided to offer the chair to Bertram Hopkinson. Their choice excited some surprise on account of his youth: but those who had been in contact with him on scientific or professional matters were found to have no misgivings.

When he returned to Cambridge to direct a department then in the full tide of development, in which some of the staff were his seniors, there were serious problems to be faced. For instance, the success of the school had made it very popular, and the question of maintaining the standard at the cost possibly of depletion of the numbers was urgent. The new professor had no hesitation on this subject. A first class in Engineering has always been a select degree: and in time a preliminary qualifying

* *Proc. London Math. Soc.*, Ser. 1, Vol. 29 (1898).

examination was instituted to keep away from the Honours school students who were not qualified to profit by the course. He was moreover not likely to undervalue the importance of a broad and solid foundation in mathematical physics, for which a University course is so well adapted.

He knew how to separate the general direction of his School from absorption in details: accordingly he had time to become influential in the work of Engineering science outside, and he promoted the progress of the science on many committees and organizations. In this way his own scientific investigations were kept in touch with the practical needs of professional life, and problems of importance came to his laboratory for his pupils to work on. Thus the rapid development of oil engines opened out many problems in the regulation of gaseous explosions: and the variety and special properties of the new kinds of steel, which have introduced so many possibilities, demanded precise investigations. In another domain, problems on impact led on to the penetration of rifle bullets and the study of explosives. A short treatise on vibrations expounded that dynamical subject on its practical side. By the numerous scientific memoirs that issued from the department its public reputation as a centre of scientific progress was maintained.

Just before the outbreak of the war he was chosen as a member of the important Advisory Committee appointed to deal with a handsome endowment, provided by the Government for the promotion of science in its application to industry. But when war broke out nothing but active service would make him content, and he withdrew from all work not connected with military problems. He had prepared himself for taking his part in the National defence by organising and inspiring as Commanding Officer the Company of Engineers attached to the University Officers' Training Corps. This work was not new to him. His father had been the leader in organising the body of Electrical Volunteers from the Engineering profession, and had been its first honorary Colonel. The son was active in following up this work, and furthered the erection of their headquarters: during the Boer War he assisted in the despatch of several contingents of this essential Corps to South Africa. He now took up a Commission in the Royal Engineers, and was assigned to experimental work and the supervision of the wireless telegraphy of the coast defences, afterwards passing on to work at the Admiralty. When the air forces began to assume an important place in the war, his special experience in the design and working of internal combustion motors marked him out for service with that arm. On the establishment of the Ministry, he was given a responsible position on the staff at Headquarters, finally with the rank of Lieutenant-Colonel, where he soon impressed the

military and Parliamentary chiefs. It belongs to the business of an air officer, while discouraging bravado, not to fear risks or to restrict enterprise; his visits of inspection were usually made by air; on one of these numerous journeys while flying alone but with another machine travelling in company, an accident due to atmospheric causes suddenly terminated his brilliant career.

The Air Council have taken the unusual and remarkable course of recording by public Resolution "their deep sense of the high and permanent value of the work done for the Flying Forces by the late Colonel Hopkinson, and their recognition of the private self-abnegation with which he devoted his great abilities and scientific attainments to the public service": and they directed that there should be communicated to the Vice-Chancellor "an expression of profound regret at his untimely death and at the loss which has thereby fallen on the University of Cambridge." Personal expressions of regret and condolence have come from the highest in the land.

He was elected F.R.S. in 1910. The C.M.G. came to him recently for public service. A few years ago he became a Professorial Fellow of King's College. On the social side his home, a centre of happiness and content, became also a centre of undergraduate interests, especially in recent years when his brother Cecil, who lived with his mother in the house adjacent, was at College. The death of this only remaining brother a short time ago, after a long illness ensuing from a severe wound received in Flanders, borne with cheerful resignation, completes this family history of sacrifice in the national service, which has left our country so much poorer. But it is only one chapter in the Cambridge record.

Bertram Hopkinson had lived long enough to leave his mark permanently on his University. His life was lost in the military defence of his country, to which he had strenuously devoted all his powers at a crisis in the national fortunes. His death at the age of 44 years is, to his colleagues and to his friends, and not less to the University, an irreparable loss.

J. L.

LORD RAYLEIGH.

LORD RAYLEIGH (President of the Society 1876–1878) died on June 30th, 1919, aged 77 years. An extended notice will appear in Vol. 19.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE DYNAMICAL EQUATIONS OF THE TIDES

Part I.—Tidal Systems with Finite Freedom.

By J. Proudman.

[Read March 9th, 1916. — Received June 6th, 1916. —
Received, in revised form, November 3rd, 1917.]

1. This is the preliminary part of an investigation in which the dynamical equations of the tides are obtained and used in Lagrange's form with an unlimited number of degrees of freedom. The transformation to this form from the ordinary hydrodynamical one is made in Part II for flat seas, and in Part III for oceans on a sphere.

The present part concerns the oscillations of a dynamical system of the same type as those provided by the tides, except that we take only a finite number of degrees of freedom. The reasoning is thereby not encumbered with questions of convergence, and this is the course which writers have usually pursued; but some of the results cannot be applied to systems with infinite freedom without further justification.

Investigations of Thomson and Tait, Lamb and Rayleigh, are brought together and extended so as to elucidate several aspects of the general theory, while tidal systems are reduced so that the coordinates can be separated into three discriminating classes. With the coordinates of the first class are associated "oscillations of the first class";* with those of

* See, for instance, Lamb, *Hydrodynamics*, 3rd ed., p. 332; 4th ed., p. 338.

the second class " oscillations of the second class " and free constant displacements; with those of the third class free steady motions.

By means of this classification tidal systems are themselves classified into systems of three types corresponding to a classification of oceans already given.*

Further results are obtained concerning limiting forms of forced motion of long period, and slowly rotating systems.

Definition of a Tidal System.

2. Let $$q_1, q_2, \ldots, q_N$$ be a set of parameters which are functions only of the time t; these we shall call the coordinates of the system. Let

$$T\{q\}$$

denote a homogeneous quadratic function of $q_1, q_2, \ldots, q_N$ with constant coefficients, which is always positive when any of the coordinates are different from zero. Then $T\{\dot{q}\}$, which we shall call the kinetic energy of the system, will be the same function of $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N$, that $T\{q\}$ is of $q_1, q_2, \ldots, q_N$. Also, let

$$V\{q\}$$

denote a homogeneous quadratic function of $q_1, q_2, \ldots, q_N$, with constant coefficients, which is never negative, though it may be zero, when all the coordinates do not vanish. This we shall call the potential energy of the system. Further, let

$$M\{q', q\}$$

denote a bilinear function of $q'_1, q'_2, \ldots, q'_N, q_1, q_2, \ldots, q_N$, with constant coefficients. Then $M\{\dot{q}, q\}$ we shall call the moment of momentum of the system. Let ω denote a parameter of which the functions T, V, M are independent; this we shall call the angular speed of rotation of the system. Again, let

$$Q_1, Q_2, \ldots, Q_N$$

be given functions only of the time t; these we shall call the components of the disturbing forces.

Then the equations

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_r} - \frac{\partial}{\partial q_r}\right) R\{\dot{q}, q\} = Q_r, \tag{1}$$

* "Limiting Forms of Long Period Tides," *Proc. London Math. Soc.*, Ser. 2, Vol. 13, p. 273 (1913).

for $r = 1, 2, \ldots, N$, where

$$R\{\dot{q}, q\} = T\{\dot{q}\} + \omega M\{\dot{q}, q\} - V\{q\}, \tag{2}$$

we shall call the equations of motion of the system. The equations of free motion are obtained by taking $Q_r = 0$, for $r = 1, 2, \ldots, N$.

This completes our definition of a tidal system, and, as thus defined, it differs from an ordinary dynamical system referred to uniformly rotating axes, in that (2) contains no term representing centrifugal action. It does correspond, however, to the systems provided by the terrestrial oceans, to the degree of approximation which consists in neglecting the centrifugal effect on g—the apparent acceleration due to gravity. But we may develop the theory of a system such as we have defined, and then when we come to applications we may allow for centrifugal action in the adopted value of g.

Take
$$2T\{q, q'\} = \sum_{r=1}^{N} q'_r \frac{\partial}{\partial q_r} T\{q\}, \tag{3}$$

$$2B\{q, q'\} = M\{q, q'\} - M\{q', q\}, \tag{4}$$

and
$$2\beta_{r,s} = \left(\frac{\partial^2}{\partial q'_r \partial q_s} - \frac{\partial^2}{\partial q_r \partial q'_s}\right) M\{q', q\}. \tag{5}$$

The constants $\beta_{r,s}$, for $r, s = 1, 2, \ldots, N$, we shall refer to as "gyroscopic coefficients." Then we shall have

$$T\{q', q\} = T\{q, q'\}, \tag{6}$$

$$B\{q', q\} = -B\{q, q'\}, \tag{7}$$

$$\beta_{s,r} = -\beta_{r,s}, \tag{8}$$

and
$$B\{q, q'\} = \sum_{r=1}^{N} \sum_{s=1}^{N} \beta_{r,s} q_r q'. \tag{9}$$

Also,
$$T\{q+q'\} = T\{q\} + 2T\{q, q'\} + T\{q'\}, \tag{10}$$

$$B\{q, q'+q''\} = B\{q, q'\} + B\{q, q''\}, \tag{11}$$

which relations may be immediately extended to the case in which q_r is the sum of any number of terms, for $r = 1, 2, \ldots, N$.

The equations of motion (1) may now be written in the form

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_r} T\{\dot{q}\} + 2\omega \sum_{s=1}^{N} \beta_{r,s} \dot{q}_s + \frac{\partial}{\partial q_r} V\{q\} = Q_r, \tag{12}$$

for $r = 1, 2, \ldots, N$.

Now it is known that independent linear combinations $p_1, p_2, \ldots p_N$ of $q_1, q_2, \ldots, q_N$ exist such that

$$\left.\begin{aligned} 2T\{\dot{q}\} &= \dot{p}_1^2+\dot{p}_2^2+\ldots+\dot{p}_N^2, \\ 2V\{q\} &= \nu_1^2 p_1^2+\nu_2^2 p_2^2+\ldots+\nu_N^2 p_N^2, \end{aligned}\right\} \tag{13}$$

$\nu_1^2, \nu_2^2, \ldots, \nu_N^2$, the "coefficients of stability", being positive or zero. The equations of motion in these coordinates $p_1, p_2, \ldots, p_N$ will be of the form

$$\ddot{p}_r+2\omega \sum_{s=1}^{N} \beta_{r,s}\dot{p}_s+\nu_r^2 p_r = P_r, \tag{14}$$

for $r = 1, 2, \ldots, N$, where $P_1, P_2, \ldots, P_N$ are the corresponding components of disturbing force, and of course the gyroscopic coefficients are different from those in (12).

When any coordinate is such that it occurs isolated in the expressions for both the kinetic and potential energies, *i.e.* when it only enters as do $p_1, p_2, \ldots, p_N$ in (13), we shall call it a "principal coordinate". When the coordinates can be separated into classes such that modes of free oscillation exist in which only the coordinates of one class vary, we shall call such classes "fundamental classes". When a fundamental class contains only one coordinate we shall call it a "fundamental coordinate". When a coordinate is at the same time principal and fundamental, we shall call it a "normal coordinate".

By the omission of centrifugal terms indicated above, we avoid the disadvantage of principal coordinates mentioned by Lamb;* for our system the transformation to such coordinates is independent of ω.

Special Tidal System.

3. We shall require to investigate a system of a very special type; it is that in which all the coefficients of stability are equal, so that, in addition to the relations of the preceding section, we have

$$V\{q\} = \nu^2 T\{q\}, \tag{15}$$

ν^2 being a positive constant.

Such a system is considered by Thomson and Tait,† who show the existence of the specially simple set of principal coordinates which is ob-

* *L.c.*, 3rd ed., p. 301; 4th ed., p. 308.

† *Natural Philosophy*, 2nd ed., Pt. 1, §§ 345 (xii)–345 (xxviii).

tained in the present section. Their method is entirely different from that here followed.

For the fundamental modes, we have, on writing in (12),

$$Q_r = 0, \quad q_r = a_r e^{i\sigma t},$$

for $r = 1, 2, \ldots, N$,

$$-\sigma^2 \frac{\partial}{\partial a_r} T\{a\} + 2\omega i\sigma \sum_{s=1}^{N} \beta_{r,s} a_s + \nu^2 \frac{\partial}{\partial a_r} T\{a\} = 0. \tag{16}$$

Take β so that

$$-\sigma^2 + 2\omega\beta\sigma + \nu^2 = 0, \tag{17}$$

and (16) becomes

$$i\beta \frac{\partial}{\partial a_r} T\{a\} + \sum_{s=1}^{N} \beta_{r,s} a_s = 0, \tag{18}$$

for $r = 1, 2, \ldots, N$.

The determinantal equation for the possible values of β is of the N-th degree in β and remains unchanged when the sign of β is changed. The roots therefore occur in equal and opposite pairs with at least one zero root when N is odd. Let β_n denote one of the possible values of β, so that in all there will be N values of n. Associated with any suffix n there will be a suffix n', such that

$$\beta_{n'} = -\beta_n,$$

and n' may be the same as n if $\beta_n = 0$.

We shall denote by $a_{r,n}$ the value of a_r (for $r = 1, 2, \ldots, N$) in the mode to which β_n refers.

Now it may be deduced from well known results that all the possible values of β are real and that sets of ratios

$$a_{1,n} : a_{2,n} : \ldots : a_{N,n}$$

may be found for each β_n, such that those corresponding to different values of n are independent, whether the corresponding values of β_n are equal or not.

We see that, corresponding to a zero value of β_n, $a_{r,n}$ may be taken real for $r = 1, 2, \ldots, N$, but that this will not be so for a non-zero value of β_n. We may take $a_{r,n}$ and $a_{r,n'}$ of such relative magnitude for $r = 1, 2, \ldots, N$ that they are conjugate complexes. We shall suppose this done; and, when $a_{r,n}$ is real for $r = 1, 2, \ldots, N$, we shall have $n' = n$.

It then appears that $T\{a_n, a_{n'}\}$ will be real and positive, so that we may take

$$2T\{a_n, a_{n'}\} = 1, \tag{19}$$

for all the N values of n. The determination of the a's is still not unique: for instance, we may multiply all the members of a complex set by any complex number of modulus unity, provided we multiply all the members of the conjugate set by the conjugate complex number. When the values of β_n are not all different the indeterminacy will be greater.

Take now $\beta = \beta_n$ in (18), multiply by $a_{r,m}$, and add for $r = 1, 2, \ldots, N$: we obtain

$$2i\beta_n T\{a_m, a_n\} + B\{a_m, a_n\} = 0. \tag{20}$$

Similarly we may obtain

$$2i\beta_m T\{a_n, a_m\} + B\{a_n, a_m\} = 0, \tag{20}'$$

and we deduce from these and (6), (7), that

$$T\{a_m, a_n\} = 0, \qquad B\{a_m, a_n\} = 0, \tag{21}$$

unless $m = n'$. The case of $\beta_m = -\beta_n$, $m \neq n'$, can be made to conform to (21) by suitable choice of the sets of a's. From (19) and (20)', we have

$$B\{a_n, a_{n'}\} = i\beta_n. \tag{22}$$

Take

$$f_n = 2T\{a_{n'}, q\}, \tag{23}$$

for all the values of n; f_n will be a real or complex linear combination of $q_1, q_2, \ldots, q_N$, according as the numbers $a_{r,n}$, for $r = 1, 2, \ldots, N$, are all real or not.

We see that f_n will be zero in every mode except that in which $q_r = Ca_{r,n}e^{i\sigma t}$, σ being an appropriate speed and C any constant, and that then $f_n = Ce^{i\sigma t}$. This shows that the set f_n, for all the values of n, is one of *fundamental coordinates*.

On solving (23) for q_r, we obtain

$$q_r = \sum_n a_{r,n} f_n, \tag{24}$$

for $r = 1, 2, \ldots, N$, $\sum_n$ denoting summation with respect to all the values of n.

From (10), (19), (21), (24), we have

$$2T\{q\} = \sum_n f_n f_{n'}. \tag{25}$$

This shows that when $n' = n$, f_n is a principal coordinate and therefore a *normal coordinate*; when $n' \neq n$, the two coordinates $f_n, f_{n'}$ occur in (25) in the form $2f_n f_{n'}$.

Multiply (12) by $a_{r,n}$ and add for $r = 1, 2, \ldots, N$; we obtain

$$2T\{a_n, \ddot{q}\} + 2\omega B\{a_n, \dot{q}\} + 2\nu^2 T\{a_n, q\} = \sum_{r=1}^{N} a_{r,n} Q_r,$$

which, on writing $$\sum_{r=1}^{N} a_{r,n} Q_r = F_n, \qquad (26)$$

for all the values of n, takes the form

$$\ddot{f}_{n'} + 2i\omega\beta_n \dot{f}_{n'} + \nu^2 f_n = F_n, \qquad (27)$$

for all the values of n, since

$$B\{a_n, \dot{q}\} = B\{a_n, a_{n'}\}\dot{f}_{n'} = i\beta_n \dot{f}_{n'},$$

by virtue of (21), (22), (24). F_n is the component of external force corresponding to f_n, but it only affects the value of $f_{n'}$.

Of course the principal coordinates are not uniquely determinate, but we may obtain a specially simple set by means of the fundamental coordinates we have discovered. Take

$$p_n = f_n,$$

when $n' = n$, and

$$\left.\begin{aligned} \sqrt{2}\, p_n &= f_{n'} + f_n, \\ \sqrt{2}\, ip_{n'} &= f_{n'} - f_n, \end{aligned}\right\} \qquad (28)^*$$

when $n' \neq n$. Then, from (23), we see that p_n, $p_{n'}$ will be real combinations of $q_1, q_2, \ldots, q_N$, for all the values of n. We deduce

$$\left.\begin{aligned} \sqrt{2}\, f_{n'} &= p_n + ip_{n'}, \\ \sqrt{2}\, f_n &= p_n - ip_{n'}, \end{aligned}\right\} \qquad (29)^*$$

so that, from (25), we obtain

$$2T\{q\} = \sum_n p_n^2, \qquad (30)$$

which shows that p_n is a *principal coordinate*, for all the values of n. From (27) we deduce that the equations of motion in these principal coordinates are

$$\ddot{p}_n - 2\omega\beta_n \dot{p}_{n'} + \nu^2 p_n = P_n, \qquad (31)$$

for all the values of n, where, when $n' = n$,

$$P_n = F_n,$$

---

* In these formulæ n and n' are not interchangeable, but the parts played by p_n, $p_{n'}$ are.

and, when $n' \neq n$,

$$\left.\begin{aligned} \sqrt{2}\, P_n &= F_n + F_{n'}, \\ \sqrt{2}\, iP_{n'} &= F_n - F_{n'}. \end{aligned}\right\} \tag{32}$$

When $\beta_n = 0$, p_n and $p_{n'}$ will be *normal coordinates;* when $\beta_n \neq 0$, they form a *fundamental class.*

We have now seen that the motion of the special system of the present section may be reduced to that of systems each of which has only one degree of freedom, when the complex fundamental coordinates are used, or to that of systems each of which has not more than two degrees of freedom, when a certain set of real principal coordinates is used.

We now consider the motion of one of these special systems of *two degrees of freedom.* Its equations of motion in principal coordinates p, p' are

$$\left.\begin{aligned} \ddot{p} - 2\omega\beta\dot{p}' + \nu^2 p &= P, \\ \ddot{p}' + 2\omega\beta\dot{p} + \nu^2 p' &= P', \end{aligned}\right\} \tag{33}$$

and in fundamental coordinates f, f' are

$$\left.\begin{aligned} \ddot{f}' + 2\omega i\beta\dot{f}' + \nu^2 f' &= F, \\ \ddot{f} - 2\omega i\beta\dot{f} + \nu^2 f &= F', \end{aligned}\right\} \tag{34}$$

the connections being

$$\sqrt{2}\, f' = p + ip', \qquad \sqrt{2}\, F = P + iP',$$

$$\sqrt{2}\, f = p - ip', \qquad \sqrt{2}\, F' = P - iP'.$$

As has already been indicated, the principal coordinates are not uniquely determinate, and in fact we may take instead of p, p',

$$p\cos\lambda - p'\sin\lambda, \qquad p\sin\lambda + p'\cos\lambda,$$

the corresponding fundamental coordinates being then

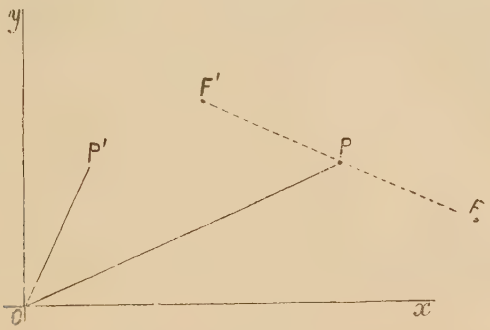
$$f'e^{i\lambda}, \qquad fe^{-i\lambda},$$

which shows that effectively these are determinate.

When we work with a time factor $e^{i\sigma t}$, p, p' are complex variables of which only the real parts are to be interpreted.

In order to examine the nature of the motion, let us take a geometrical

illustration. On an Argand diagram let P and P' represent p and p' respectively, so that the projections on Ox of OP and OP' will represent



the parts to be interpreted. Let F and F' represent respectively $\sqrt{2}f$ and $\sqrt{2}f'$.

For the free modes in which f' is zero, OP' is equal to OP and a right angle in advance, and the motions are represented by rotating the figure forwards with either of the speeds given by

$$-\sigma^2+2\omega\beta\sigma+\nu^2=0. \qquad (35)$$

For the modes in which f is zero, OP' is equal to OP and a right angle behind, while the motions are represented by rotating the figure backwards with either of the speeds given by (35). Now, as regards projections on Ox, we see that we get the same modes whether we take f' or f zero.

Taking $f'=0$, we have for the two possible speeds

$$\omega\beta\pm(\omega^2\beta^2+\nu^2)^{\frac{1}{2}},$$

and when ν is not zero these cannot be equal. When $\omega\beta$ is small, so that we may neglect its fourth and higher powers, and ν is not small, these may be written

$$\pm\nu+\omega\beta\pm\tfrac{1}{2}\frac{\omega^2\beta^2}{\nu},$$

the two upper signs or the two lower signs in the ambiguities being taken together. When $\nu=0$ and $\beta\neq 0$, they become, for all values of ω,

$$0, \quad 2\omega\beta,$$

giving a *constant displacement* (not steady motion) and one oscillation with a speed proportional to the first power of ω. When $\nu=0$ and also $\beta=0$, we have two *steady motions*.

For forced periodic motion of speed σ, we have

$$(-\sigma^2-2\omega\beta\sigma+\nu^2)f' = F,$$
$$(-\sigma^2+2\omega\beta\sigma+\nu^2)f = F',$$

giving

$$\left.\begin{aligned} f &= -\frac{F'}{(\sigma-\sigma_1)(\sigma-\sigma_2)}, \\ f' &= -\frac{F}{(\sigma+\sigma_1)(\sigma+\sigma_2)}, \end{aligned}\right\} \tag{36}$$

where σ_1, σ_2 are the two free speeds given by (35).

Reduction of General System.

4. We now return to the general system of § 2 and suppose that principal coordinates are used. Let us separate these coordinates into classes, so that all the members of one class have equal coefficients of stability, and with different classes are associated different coefficients of stability. Consider the auxiliary system formed by suppressing all the coordinates except those belonging to one of the classes we have just defined. Such a system is of the special type of the preceding section, and its coordinates may be transformed into the fundamenta ones there shown to exist. The aggregate of the formulæ of transformation for all the different classes may now be used to transform the coordinates of the whole original system, and we shall suppose this done.

We shall now have

$$2T\{q\} = \sum_{r=1}^{N} f_r f_{r'}, \qquad 2V\{q\} = \sum_{r=1}^{N} \nu_r^2 f_r f_{r'}, \tag{37}$$

while the equations of motion become

$$\ddot{f}_{r'}+2\omega i\beta_r \dot{f}_{r'}+\nu_r^2 f_{r'}+2\omega \sum_{s=1}^{N} \beta_{r,s}\dot{f}_s = F_r, \tag{38}$$

for $r = 1, 2, \ldots, N$, where now

$$\beta_{r,s} = 0,$$

if $\nu_s^2 = \nu_r^2$, and what would be $\beta_{r,r'}$ in the notation of § 2 is now written $i\beta_r$ in conformity with the preceding section.

We have $\beta_{r'} = -\beta_r$, and, when $r' = r$, f_r is a principal coordinate; but no coordinate is necessarily a fundamental one for the whole system. When only one coordinate has a certain value for its coefficient of stability,

it will be identical with the corresponding original principal coordinate, and for it $\beta_r = 0$.

If we now use the further transformation of p. 7, we obtain a special set of principal coordinates which are real linear combinations of the original ones. On adding the equation written in (38) to that obtained by interchanging r and r', we see that

$$(\beta_{r,s}+\beta_{r',s})+(\beta_{r,s'}+\beta_{r',s'})$$

is real, and

$$(\beta_{r,s}+\beta_{r',s})-(\beta_{r,s'}+\beta_{r',s'})$$

is purely imaginary, from the necessary reality of the coefficients of $\dot{p}_s$ and $\dot{p}_{s'}$. It follows that

$$\beta_{r,s}+\beta_{r',s}, \quad \beta_{r,s'}+\beta_{r',s'}$$

are conjugate complexes. Similarly, by subtracting the same two equations, we see that

$$(\beta_{r,s}-\beta_{r',s})+(\beta_{r,s}-\beta_{r',s'})$$

is purely imaginary, and

$$(\beta_{r,s}-\beta_{r',s})-(\beta_{r,s'}-\beta_{r',s'})$$

is real, from which it follows that

$$\beta_{r,s}-\beta_{r',s}, \quad \beta_{r,s'}-\beta_{r,s'}$$

are conjugate complexes. Putting together the two results we see that

$$\beta_{r,s}, \ \beta_{r',s'}; \quad \beta_{r',s}, \ \beta_{r,s'}$$

are two pairs of conjugate complexes. The conjugacy of one of these pairs would follow from that of the other by interchanging r and r' or s and s'.

We shall now separate the coordinates f_r of (38) into three classes. We shall say that f_r is a "coordinate of the first class" if the corresponding coefficient of stability is not zero; we shall say that it is a "coordinate of the second class" if the corresponding coefficient of stability is zero but the corresponding value of β_r is not zero; we shall say that it is a "coordinate of the third class" if the corresponding values of the coefficient of stability and β_r are zero.

The coordinates of the second class occur in associated pairs; in no case have we $n' = n$. For any coordinate of the third class we may have $n' = n$, in which case the coordinate will be a principal one.

When a system contains no coordinates of the third class, we shall say that it is of "Type I"; when it contains coordinates of the third class but none of the second class, we shall say that it is of "Type II"; when it contains coordinates of both the second and third classes, we shall say that it is of "Type III".

We shall always suppose that the components of disturbing force corresponding to coordinates of the second and third classes are zero, in order that our system may correspond to an actual tidal system.

Let us now change the notation so that the coordinates of the first class are f_r for

$$r = 1, 2, \ldots, N_1,$$

those of the second class are f_{-r} for

$$r = 1, 2, \ldots, N_2,$$

while those of the third class are f_{-r} for

$$r = N_2+1,\ N_2+2,\ \ldots,\ N_2+N_3.$$

Then we shall have

$$T\{q\} = T_1\{q\}+T_2\{q\}+T_3\{q\}, \tag{39}$$

where

$$2T_1\{q\} = \sum_{r=1}^{N_1} f_r f_{r'}, \quad T_2\{q\} = \sum_{r=1}^{N_2} f_{-r} f_{-r'}, \quad T_3\{q\} = \sum_{r=N_2+1}^{N_2+N_3} f_{-r} f_{-r'}, \tag{40}$$

while

$$V\{q\} = \sum_{r=1}^{N_1} \nu_r^2 f_r f_{r'}. \tag{41}$$

The equations of motion will be

$$\ddot{f}_{r'}+2\omega i\beta_r \dot{f}_{r'}+2\omega \sum_{s=1}^{N_1} \beta_{r,s}\dot{f}_s+\nu_r^2 f_{r'}+2\omega \sum_{s=1}^{N_2+N_3} \beta_{r,-s}\dot{f}_{-s} = F_r, \tag{42}$$

for $r = 1, 2, \ldots, N_1$, and

$$\ddot{f}_{-r'}+2\omega i\beta_{-r}\dot{f}_{-r'}+2\omega \sum_{s=1}^{N_1} \beta_{-r,s}\dot{f}_s = 0, \tag{43}$$

for $r = 1, 2, \ldots, N_2+N_3$.

We will now eliminate from the equations of motion the coordinates of the second and third classes, when the motion is periodic.

Let

$$q_1,\ q_2,\ \ldots,\ q_{N_1}$$

be any independent linear combinations of the coordinates of the first

class. Then if σ be the speed of the motion we shall have, instead of (42) and (43),

$$-\sigma^2 \frac{\partial}{\partial q_r} T_1\{q\} + 2\omega i\sigma \sum_{s=1}^{N_1} \beta_{r,s} q_s + \frac{\partial}{\partial q_r} V\{q\} + 2\omega \sum_{s=1}^{N_2+N_3} \beta_{r,-s} \dot{f}_{-s} = Q_r, \quad (44)$$

for $r = 1, 2, \ldots, N_1$, and

$$\dot{f}_{-r'} + 2\omega i \beta_{-r} f_{-r'} + 2\omega \sum_{s=1}^{N_1} \beta_{-r,s} q_s = 0, \quad (45)$$

for $r = 1, 2, \ldots, N_2+N_3$. Of course the gyroscopic coefficients are different from those in (42) and (43), and we now easily see that, if $q_1, q_2, \ldots, q_N$ are real combinations of the original coordinates, $\beta_{-r,s}$ and $\beta_{-r',s}$ will be conjugate complexes.

Now

$$2\omega \sum_{t=1}^{N_2+N_3} \beta_{r,-t} \dot{f}_{-t}$$

$$= 4\omega^2 \sum_{t=1}^{N_2+N_3} \frac{\beta_{-t,r}}{1-\frac{2\omega}{\sigma}\beta_{-t}} \sum_{s=1}^{N_1} \beta_{-t',s} q_s$$

$$= 2\omega^2 \sum_{t=1}^{N_2+N_3} \sum_{s=1}^{N_1} \left\{ \frac{\beta_{-t,r}\beta_{-t',s}}{1-\frac{2\omega}{\sigma}\beta_{-t}} + \frac{\beta_{-t',r}\beta_{-t,s}}{1+\frac{2\omega}{\sigma}\beta_{-t}} \right\} q_s$$

$$= 2\omega^2 \sum_{t=1}^{N_2+N_3} \frac{1}{1-\frac{4\omega^2}{\sigma^2}\beta_{-t}^2} \sum_{s=1}^{N_1} \Big\{ (\beta_{-t,r}\beta_{-t',s} + \beta_{-t',r}\beta_{-t,s}) + \frac{2\omega}{\sigma}\beta_{-t}(\beta_{-t,r}\beta_{-t',s} - \beta_{-t',r}\beta_{-t,s}) \Big\} q_s$$

$$= 4\omega^2 \frac{\partial}{\partial q_r} K_\sigma\{q\} + 2\omega i\sigma \sum_{s=1}^{N_1} \beta'_{r,s} q_s,$$

where
$$2K_\sigma\{q\} = \sum_{t=1}^{N_2+N_3} \frac{1}{1-\frac{4\omega^2}{\sigma^2}\beta_{-t}^2} \left(\sum_{s=1}^{N_1} \beta_{-t,s} q_s \right) \left(\sum_{s=1}^{N_1} \beta_{-t',s} q_s \right), \quad (46)$$

and
$$i\beta'_{r,s} = \sum_{t=1}^{N_2} \frac{4\omega^2 \beta_{-t}}{\sigma^2 - 4\omega^2\beta_{-t}^2} \beta_{-t,r}\beta_{-t',s}, \quad (47)$$

for $r, s = 1, 2, \ldots, N_1$.

$K_\sigma\{q\}$ is a homogeneous quadratic function of $q_1, q_2, \ldots, q_{N_1}$, and $\beta'_{r,s}$

is a constant; both will be real if $q_1, q_2, \ldots, q_N$ are real combinations of the original principal coordinates, $K_\sigma\{q\}$ being never negative if $q_1, q_2, \ldots, q_N$ are not all zero. We see that

$$\beta'_{s,r} = -\beta'_{r,s}, \tag{48}$$

for $r, s = 1, 2, \ldots, N_1$; and, as $\sigma \to 0$,

$$K_\sigma\{q\} \to K_0\{q\},$$

where

$$\begin{aligned} 4\omega^2 K_0\{q\} &= 2\omega^2 \sum_{t=N_2+1}^{N_2+N_3} \left(\sum_{s=1}^{N_1} \beta_{-t,s} q_s\right)\left(\sum_{s=1}^{N_1} \beta_{-t',s} q_s\right) \\ &= \tfrac{1}{2} \sum_{t=N_2+1}^{N_2+N_3} \dot{f}_{-t} \dot{f}_{-t'} \\ &= T_3\{\dot{q}\}. \end{aligned} \tag{49}$$

The equations (44) may therefore be written

$$-\sigma^2 \frac{\partial}{\partial q_r} T_1\{q\} + 2\omega i\sigma \sum_{s=1}^{N_1} (\beta_{r,s} + \beta'_{r,s}) q_s + \frac{\partial}{\partial q_r}(V\{q\} + 4\omega^2 K_\sigma\{q\}) = Q_r, \tag{50}$$

for $r = 1, 2, \ldots, N_1$. We have here N_1 equations in the N_1 coordinates $q_1, q_2, \ldots, q_{N_1}$.

For a system of Type II, $N_2 = 0$ and $\beta'_{r,s} = 0$ for $r, s = 1, 2, \ldots, N_1$, while $K_\sigma\{q\} = K_0\{q\}$.

Multiply (50) by q_r and add for $r = 1, 2, \ldots, N_1$; we obtain, after dividing by 2,

$$-\sigma^2 T_1\{q\} + V\{q\} + 4\omega^2 K_\sigma\{q\} = \tfrac{1}{2} \sum_{r=1}^{N_1} Q_r q_r, \tag{51}$$

by virtue of Euler's theorem for homogeneous functions and (8), (48). This we may call the energy integral of the equations (50).

Constant Displacements, Steady Motions and Limiting Forms.

5. In the equations (42) let us suppose that $F_1, F_2, \ldots, F_{N_1}$ are constants.

If all the coordinates are of the first class, the solution of (42) will be

$$f_{r'} = \frac{F_r}{\nu_r^2} = \bar{f}_{r'}, \tag{52}$$

say, for $r = 1, 2, \ldots, N_1$. The system thus undergoes a constant displacement which vanishes with all the forces; this we shall call the "equilibrium solution."

If f_{-r} is a coordinate of the second class, $\dot{f}_{-r}$ will be zero, but f_{-r} may be any constant and its value will not affect the other coordinates.

If f_{-r} is a coordinate of the third class, $\dot{f}_{-r}$ may be any constant, but its value will affect the coordinates of the first class.

For a system of Type I, the coordinates of the second class may have any constant values, and (52) holds for those of the first class. The system thus undergoes a constant displacement and the coordinates of the first class have their equilibrium values. Free or forced steady motions are impossible.

A system of Type II undergoes a steady motion in the coordinates of the third class, and a constant displacement in the coordinates of the first class, but this constant displacement will not in general take the equilibrium form unless the steady motion vanishes. An independent free steady motion is possible corresponding to each coordinate of the third class.

A system of Type III undergoes a steady motion in the coordinates of the third class, and a constant displacement in those of the other classes. The coordinates of the first class do not in general take their equilibrium values unless the steady motion vanishes. Free constant displacements and free steady motions are possible.

Now let us consider the limiting form of forced periodic motion as the period tends to become infinite. The existence of unique limiting values of the coordinates of the first class, and of the speeds of the other coordinates, can be deduced from the general theory of forced oscillations of tidal systems with finite freedom.*

Let us utilise the equations (45) and (50). We see that the limiting forms of $q_1, q_2, \ldots, q_{N_1}$ will be given by

$$\frac{\partial}{\partial q_r}\left(V\{q\} + 4\omega^2 K_0\{q\}\right) = Q_r, \tag{53}$$

for $r = 1, 2, \ldots, N_1$, where $Q_1, Q_2, \ldots, Q_{N_1}$ are the constants to which the disturbing forces tend. The constant coordinates of the second class and the constant speeds of those of the third class will be given by (45). The kinetic energy of the limiting steady motion will be

$$T_3\{\dot{q}\} = 4\omega^2 K_0\{q\}.$$

* See Poincaré, *Leçons de mécanique céleste*, t. 3, c. 1.

The corresponding equilibrium values of $q_1, q_2, \ldots, q_{N_1}$ are given by

$$\frac{\partial}{\partial \bar{q}_r} V\{\bar{q}\} = Q_r, \tag{54}$$

for $r = 1, 2, \ldots, N_1$.

Now let us consider, for a moment, the tidal system in which $V\{\dot{q}\}$ is the kinetic energy and $K_0\{q\}$ the potential energy. Principal coordinates will exist which are linear functions of $q_1, q_2, \ldots, q_{N_1}$; let these be $\theta_1, \theta_2, \ldots, \theta_{N_1}$, so chosen that

$$\left.\begin{aligned} 2V\{q\} &= \theta_1^2+\theta_2^2+\ldots+\theta_{N_1}^2, \\ 2K_0\{q\} &= \gamma_1^2\theta_1^2+\gamma_2^2\theta_2^2+\ldots+\gamma_{N_1}^2\theta_{N_1}^2, \end{aligned}\right\} \tag{55}$$

the constants γ being real. Now we may use these coordinates instead of $q_1, q_2, \ldots, q_{N_1}$ in (53) and (54); let us do this and suppose the corresponding components of disturbing force are $\Theta_1, \Theta_2, \ldots, \Theta_{N_1}$. We obtain

$$(1+4\omega^2\gamma_r^2)\,\theta_r = \Theta_r,$$

$$\bar{\theta}_r = \Theta_r,$$

giving

$$(1+4\omega^2\gamma_r^2)\,\theta_r = \bar{\theta}_r, \tag{56}$$

for $r = 1, 2, \ldots, N_1$. This shows that θ_r is never greater than $\bar{\theta}_r$, and, if not zero, is only equal to it when γ_r is zero.

From (56) we have

$$\bar{\theta}_r^2 = \theta_r^2+(\bar{\theta}_r-\theta)^2+8\omega^2\gamma_r^2\theta_r^2,$$

on adding which for $r = 1, 2, \ldots, N_1$, and dividing by 2, we obtain

$$\begin{aligned} V\{\bar{q}\} &= V\{q\}+V\{\bar{q}-q\}+8\omega^2K_0\{q\} \\ &= V\{q\}+V\{\bar{q}-q\}+2T_3\{\dot{q}\}, \end{aligned} \tag{57}$$

showing that the potential energy of the equilibrium form exceeds that of the limiting form by that of the difference between the two forms increased by twice the kinetic energy of the limiting form.

For a system of Type I, the limiting form is one of rest, $K_0\{q\}$ is identically zero, and the limiting values of $q_1, q_2, \ldots, q_{N_1}$ are the equilibrium ones. For systems of Types II and III, $K_0\{q\}$ is not identically zero, and the limiting forms are not in general the same as the equili-

brium forms. A necessary and sufficient condition that the limiting form shall be the same as the equilibrium form is that $\Theta_r = 0$ for every value of r for which γ_r does not vanish.

These results are analogous to those obtained for hydrodynamical systems in the paper quoted on p. 2.

Attention was first directed to steady motions and non-equilibrium limiting forms by Lamb*; he considered in particular a system of Type II with two degrees of freedom. Later Hough† considered systems of Types I and II with two degrees of freedom, but took disturbing forces corresponding to his coordinates of the second class. Lord Rayleigh‡ afterwards considered a system of Type I with three degrees of freedom.

Free Motion of Slowly Rotating System.

6. Lord Rayleigh has discussed a tidal system with ω small, and has found the effect of a small rotation on free modes which correspond respectively to only one and two coefficients of stability having a certain value.

Now we may have more than two coefficients of stability equal, and in fact, in actual tidal systems an infinite number of them are zero. We shall now apply Rayleigh's method to the general case. It will appear that no serious complication arises, however many coefficients of stability may be equal, so long as the values of β_r are distinct.

Let us use the notation of (38) and suppose that

$$\nu_1 = \nu_2 = \ldots = \nu_M = \nu,$$

and

$$\beta_1 = \beta_2 = \ldots = \beta_{M'} = \beta,$$

where $M' \leqslant M$. Then, the motion being periodic with speed σ, which is approximately $\pm\nu$, we have

$$(-\sigma^2 + 2\omega\beta\sigma + \nu^2) f_r + 2\omega i\sigma \sum_{s=M+1}^{N} \beta_{r,s} f_s = 0, \qquad (58)$$

---

* *Hydrodynamics*, 2nd ed., pp. 330, 334, 347, 353 (1895).

† "On the Application of Harmonic Analysis to the Dynamical Theory of the Tides, Part I", *Phil. Trans. R. S.* (A), Vol. 189, p. 246 (1897).

‡ "Note on the Theory of the Fortnightly Tide", *Phil. Mag.* (6), Vol. 5, p. 136 (1903), (*Sci. Papers*, Vol. 5, p. 84).

for $r = 1, 2, \ldots, M'$, and

$$(-\sigma^2+2\omega\beta_r\sigma+\nu^2) f_r+2\omega i\sigma \sum_{s=M+1}^{N} \beta_{r',s} f_s = 0, \tag{59}$$

for $r = M'+1, M'+2, \ldots, M$. We have also

$$(-\sigma^2+\nu_r^2) f_r+2\omega i\sigma \sum_{s=1}^{M'} \beta_{r',s} f_s = 0, \tag{60}$$

for $r = M+1, M+2, \ldots, N$, on neglecting the second and higher powers of ω. From (60) we obtain

$$f_r = \frac{2\omega i\sigma}{\sigma^2-\nu_r^2} \sum_{s=1}^{M'} \beta_{r',s} f_s, \tag{61}$$

for $r = M+1, M+2, \ldots, N$, and then from (59) on writing

$$-\sigma^2+2\omega\beta\sigma+\nu^2 = 0,$$

we obtain

$$f_r = \frac{2\omega\sigma}{\beta_r-\beta} \sum_{s=M+1}^{N} \frac{1}{\sigma^2-\nu_s^2} \sum_{t=1}^{M'} \beta_{s',t} f_t, \tag{62}$$

for $r = M'+1, M'+2, \ldots, M$. Substituting from (61) into (58), we obtain

$$(-\sigma^2+2\omega\beta\sigma+\nu^2) f_r-4\omega^2\sigma^2 \sum_{s=M+1}^{N} \frac{\beta_{r',s}}{\sigma^2-\nu_s^2} \sum_{t=1}^{M'} \beta_{s',t} f_t = 0, \tag{63}$$

for $r = 1, 2, \ldots, M'$. In (63) we have M' equations for the ratios

$$f_1 : f_2 : \ldots : f_{M'};$$

the determinantal equation for the possible values of σ will have $2M'$ roots.

If $\nu \neq 0$, we may put $\sigma^2 = \nu^2$ in the right-hand part of (63), and $\sigma = \pm\nu+\omega\beta$ in the left-hand part; this gives

$$(-\sigma^2 \pm 2\omega\beta\nu+2\omega^2\beta^2+\nu^2) f_r-4\omega^2 \sum_{s=M+1}^{N} \frac{\beta_{r',s}}{1-\nu_s^2/\nu^2} \sum_{t=1}^{M'} \beta_{s',t} f_t = 0, \tag{64}$$

for $r = 1, 2, \ldots, M'$.

If $\nu = 0$, M' roots of (63) are zero, and we may put $\sigma = 2\omega\beta$ in the right-hand side; this gives, on reverting to the notation of (40),

$$(-\sigma+2\omega\beta) f_{-r}+8\omega^3\beta \sum_{s=1}^{N_1} \frac{\beta_{-r',s}}{\nu_s^2} \sum_{t=1}^{M'} \beta_{s',-} f_{-t} = 0, \tag{65}$$

for $r = 1, 2, \ldots, M'$. We have also from (61)

$$f_r = -i\frac{4\omega^2\beta}{\nu_r^2}\sum_{s=1}^{M'}\beta_{r',\,-s}f_{-s}, \tag{66}$$

for $r = 1, 2, \ldots, N_1$, and from (62)

$$f_{-r} = \frac{4\omega^2\beta}{\beta-\beta_{-r}}\sum_{s=1}^{N_1}\frac{1}{\nu_s^2}\sum_{t=1}^{M'}\beta_{s',\,-t}f_{-t}, \tag{67}$$

for $r = M'+1, M'+2, \ldots, M$.

If $M' = 1$, we deduce from (64)

$$\sigma = \pm\nu+\omega\beta-\tfrac{1}{2}\frac{\omega^2\beta^2}{\nu} \pm \frac{\omega^2\beta^2}{\nu} \mp \frac{2\omega^2}{\nu}\sum_{s=M+1}^{N}\frac{\beta_{1',\,s}\beta_{s',\,1}}{1-\nu_s^2/\nu^2}, \tag{68}$$

where the three upper signs or the three lower signs are to be taken; from (65) we obtain

$$\sigma = 2\omega\beta+8\omega^3\beta\sum_{s=1}^{N_1}\frac{\beta_{-1',\,s}\beta_{s',\,-1}}{\nu_s^2}. \tag{69}$$

Now for a moment let us consider the general tidal system in which ω is not small. The modes and speeds of oscillation are functions of ω and vary continuously with ω. When ω becomes small, their principal parts take the forms we have been investigating in the present section. We have seen that the principal part of the speed cannot contain a power of ω higher than the first. When the principal part of the speed does not involve ω, we shall call the oscillation one of the "first class" for all values of ω; in this case the principal part of the motion for ω small involves coordinates of the first class only. When the principal part of the speed contains ω as a factor we shall say that the oscillation is of the "second class" for all values of ω; in this case the principal part of the motion for ω small involves coordinates of the second class only. We have already seen that with a coordinate of the second class we have associated a free constant displacement for all values of ω, and that with a coordinate of the third class we have associated a free steady motion for all values of ω.

Oscillations of the second class occur in systems of Types I and III, but not in systems of Type II.

The terms "oscillations of the first and second classes" were first used by Hough* in reference to tides in an ocean covering a rotating

* "On the Application of Harmonic Analysis . . ., Part II", *Phil. Trans. R. S.*, (A), Vol. 191, p. 159 (1897).

globe. The existence of the two kinds had already been discovered by Lamb* for a particular flat circular sea of variable depth. In order to indicate the nature of the oscillations of the second class Lamb later† considered a system with two degrees of freedom like our tidal systems, except that ν_2 contained ω as a factor. This, which is essential in Lamb's case, means that ν_2 was derived entirely from centrifugal action. Now in the hydrodynamical cases the centrifugal terms had been neglected, so that it appears that oscillations of the second class can arise in a way other than that indicated by Lamb's case of two degrees of freedom. How this is so we have seen in the present section.

---

* *Hydrodynamics*, 2nd ed., p. 343 (1895).

† *Ibid.*, 3rd ed., p. 301 (1906); 4th ed., p. 308 (1916).

ON THE DYNAMICAL EQUATIONS OF THE TIDES

Part II.—Flat Seas.

By J. Proudman.

[Read March 9th, 1916.]

1. This part is concerned with flat rotating seas of any natural laws of shape and depth. The main result is a transformation of the differential equations of motion into a set of the type considered in Part I, except that now we have an unlimited number of degrees of freedom.

The algebraic linear equations which result from this transformation can be explicitly constructed in a number of cases for which the tidal problem has not yet been solved. In a future part it is hoped to consider the application to some of these cases, but in the present part the solution is only discussed in cases where the depth is a function only of the distance from a fixed point, and the boundaries are circles having this point as centre. This class includes all the cases of flat rotating seas for which the tidal problem has hitherto been completely solved; for each there exists an infinite set of fundamental classes of coordinates.

General Differential Equations.

2. We suppose that the bottom of the sea has everywhere a definite tangent plane, and that at the boundary line of the free surface the banks are not horizontal. We suppose also that the boundary line has a definite tangent at any point on it with the possible exception of a finite number of them.

Let h denote the depth at any point below the plane of the free surface in equilibrium, and, using rectangular Cartesian axes in this plane, let l, m denote the direction cosines of any outward drawn normal to the boundary. By $\partial/\partial n$ we shall denote differentiation along a normal to the boundary in the direction towards the boundary from the sea, and by $\partial/\partial s$ differentiation along the positive direction of the arc of the boundary.

We shall often say that one function tends to another at the boundary,

and by this we shall mean that their difference tends to zero as the point at which they are evaluated tends to the boundary along every normal that exists.

Unless otherwise stated, all surface integrals will be taken over the whole area of the surface of the sea, and all line integrals positively round the complete boundary, though, when the integrand of a line integral does not exist on the boundary, the limit as the boundary is approached along the normal is to be used.

Let ξ, η denote the components of the horizontal displacement of a particle of water, and ζ the elevation of the free surface at a point whose coordinates are x, y. Then ξ, η, ζ will be functions only of x, y and the time t, and the equation of continuity gives

$$\frac{\partial}{\partial x}(h\xi)+\frac{\partial}{\partial y}(h\eta)+\zeta = 0, \tag{1}$$

over the whole sea. We have also

$$h(l\xi+m\eta) \to 0, \tag{2}$$

at the boundary, which with (1) gives

$$\iint \zeta\, dS = 0, \tag{3}$$

by the aid of Green's theorem.

The kinetic energy T of the relative motion is given by

$$\frac{2T}{\rho} = \iint h(\dot{\xi}^2+\dot{\eta}^2)\, dS, \tag{4}$$

where ρ denotes the density of the water, supposed uniform; the potential energy V is given by

$$\frac{2V}{\rho} = g\iint \zeta^2\, dS, \tag{5}$$

where g denotes the acceleration due to gravity, supposed uniform; the bilinear part M of the moment of momentum of the relative motion about any vertical axis, is given by

$$\frac{M}{\rho} = \iint h(\xi\dot{\eta}-\eta\dot{\xi})\, dS. \tag{6}$$

If the sea is rotating with constant angular velocity ω about a vertical axis, and if Ω denotes the potential of the disturbing forces, the dynamical

equations of motion* are

$$\left.\begin{aligned} \ddot{\xi}-2\omega\dot{\eta} &= -\frac{\partial}{\partial x}(g\zeta+\Omega), \\ \ddot{\eta}+2\omega\dot{\xi} &= -\frac{\partial}{\partial y}(g\zeta+\Omega), \end{aligned}\right\} \tag{7}$$

to be satisfied over the whole sea.

We will first prove a lemma of which we shall make considerable use. Let X, Y be two functions which are differentiable over the sea. Take a function P such that

$$\frac{\partial}{\partial x}\left(h\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial P}{\partial y}\right) = -\left\{\frac{\partial}{\partial x}(hX)+\frac{\partial}{\partial y}(hY)\right\}, \tag{8}$$

over the sea, with

$$h\frac{\partial P}{\partial n} \rightarrow -h(lX+mY), \tag{9}$$

at the boundary; this determines P except to an additive constant. Take also a function Q such that

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial Q}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial Q}{\partial y}\right) = \frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}, \tag{10}$$

over the sea, with

$$Q/h \rightarrow 0, \tag{11}$$

at the boundary; this determines Q uniquely. We can now prove that

$$\left.\begin{aligned} X &= -\frac{\partial P}{\partial x}-\frac{1}{h}\frac{\partial Q}{\partial y}, \\ Y &= -\frac{\partial P}{\partial y}+\frac{1}{h}\frac{\partial Q}{\partial x}, \end{aligned}\right\} \tag{12}$$

over the sea. For this purpose take

$$X' = X+\frac{\partial P}{\partial x}+\frac{1}{h}\frac{\partial Q}{\partial y},$$

$$Y' = Y+\frac{\partial P}{\partial y}-\frac{1}{h}\frac{\partial Q}{\partial x},$$

and we have

$$\frac{\partial Y'}{\partial x}-\frac{\partial X'}{\partial y} = 0,$$

* See, for instance, Lamb, *Hydrodynamics*, 3rd ed., p. 302; 4th ed., p. 309.

over the sea, showing that a function P' exists such that

$$X' = -\frac{\partial P'}{\partial x}, \qquad Y' = -\frac{\partial P'}{\partial y}.$$

We then have $$\frac{\partial}{\partial x}\left(h\frac{\partial P'}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial P'}{\partial y}\right)=0,$$

over the sea, with $$h\frac{\partial P'}{\partial n}\to 0,$$

at the boundary, which shows that P' is a constant over the sea. It follows that X' and Y' are zero and therefore that (12) is true.

In this lemma take

$$X=\xi, \qquad Y=\eta, \qquad P=\phi, \qquad Q=\psi,$$

so that we have $$\frac{\partial}{\partial x}\left(h\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\phi}{\partial y}\right)=\xi, \tag{13}$$

over the sea, with $$h\frac{\partial\phi}{\partial n}\to 0, \tag{14}$$

at the boundary,

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial\psi}{\partial y}\right)=\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}, \tag{15}$$

over the sea, with $$\psi/h\to 0, \tag{16}$$

at the boundary, and

$$\left.\begin{aligned}\xi &= -\frac{\partial\phi}{\partial x}-\frac{1}{h}\frac{\partial\psi}{\partial y},\\ \eta &= -\frac{\partial\phi}{\partial y}+\frac{1}{h}\frac{\partial\psi}{\partial x},\end{aligned}\right\} \tag{17}$$

over the sea.

Also, in the lemma, take

$$X=\eta, \qquad Y=-\xi, \qquad P=\phi', \qquad Q=\psi',$$

so that we have

$$\frac{\partial}{\partial x}\left(h\frac{\partial\phi'}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\phi'}{\partial y}\right)=-\left\{\frac{\partial}{\partial x}(h\eta)-\frac{\partial}{\partial y}(h\xi)\right\}, \tag{18}$$

over the sea, with $$h\frac{\partial\phi'}{\partial n}\to -h(l\eta-m\xi), \tag{19}$$

at the boundary,

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial\psi'}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial\psi'}{\partial y}\right)=-\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right), \tag{20}$$

over the sea, with

$$\psi'/h\to 0, \tag{21}$$

at the boundary, and

$$\left.\begin{aligned}\eta&=-\frac{\partial\phi'}{\partial x}-\frac{1}{h}\frac{\partial\psi'}{\partial y},\\ -\xi&=-\frac{\partial\phi'}{\partial y}+\frac{1}{h}\frac{\partial\psi'}{\partial x},\end{aligned}\right\} \tag{22}$$

over the sea.

Again, in the lemma, take

$$X=\ddot{\xi}-2\omega\dot{\eta}+\frac{\partial}{\partial x}(g\zeta+\Omega),$$

$$Y=\ddot{\eta}+2\omega\dot{\xi}+\frac{\partial}{\partial y}(g\zeta+\Omega);$$

then

$$P=\ddot{\phi}-2\omega\dot{\phi}'-(g\zeta+\Omega),$$

$$Q=\ddot{\psi}-2\omega\dot{\psi}',$$

will satisfy (8)–(12) by virtue of (13)–(22). We thus see that the equations (7) are equivalent to

$$\left.\begin{aligned}\ddot{\phi}-2\omega\dot{\phi}'-(g\zeta+\Omega)&=\text{constant},\\ \ddot{\psi}-2\omega\dot{\psi}'&=0,\end{aligned}\right\} \tag{23}$$

so that we may take these* as the equations of the tidal problem instead of (2) and (7).

Analytical Basis of Transformation.

3. Consider a solution of the differential equation

$$\frac{\partial}{\partial x}\left(h\frac{\partial\phi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\phi_r}{\partial y}\right)+\lambda_r\phi_r=0, \tag{24}$$

---

* Cf. Love, "Notes on the Dynamical Theory of the Tides", *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 310, 311 (1913).

over the sea, λ_r being a constant, with the condition

$$h\frac{\partial \phi_r}{\partial n} \to 0, \tag{25}$$

at the boundary. We have, on using Green's theorem,

$$\begin{aligned}\lambda_r \iint \phi_r dS &= -\iint \left\{\frac{\partial}{\partial x}\left(h\frac{\partial \phi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \phi_r}{\partial y}\right)\right\} dS \\ &= -\int h\frac{\partial \phi_r}{\partial n} ds \\ &= 0, \end{aligned} \tag{26}$$

and

$$\begin{aligned}\lambda_r \iint \phi_r^2 dS &= -\iint \phi_r \left\{\frac{\partial}{\partial x}\left(h\frac{\partial \phi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \phi_r}{\partial y}\right)\right\} dS \\ &= \iint h\left\{\left(\frac{\partial \phi_r}{\partial x}\right)^2+\left(\frac{\partial \phi_r}{\partial y}\right)^2\right\} dS, \end{aligned} \tag{27}$$

showing that λ_r is positive.

Now suppose that ϕ_r and ϕ_s are solutions of (24) and (25) corresponding respectively to unequal constants λ_r and λ_s.* We have, again using Green's theorem,

$$\begin{aligned}\lambda_r \iint \phi_r \phi_s dS &= -\iint \phi_s \left\{\frac{\partial}{\partial x}\left(h\frac{\partial \phi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \phi_r}{\partial y}\right)\right\} dS \\ &= \iint h\left(\frac{\partial \phi_r}{\partial x}\frac{\partial \phi_s}{\partial x}+\frac{\partial \phi_r}{\partial y}\frac{\partial \phi_s}{\partial y}\right) dS \\ &= \lambda_s \iint \phi_r \phi_s dS \\ &= 0. \end{aligned} \tag{28}$$

When several linearly independent solutions of (24) and (25) exist corresponding to the same constant λ_r, we may choose them so that (28) is satisfied for each pair.

We shall require further properties of the functions we are now considering, and to obtain them we shall quote from the theory of integral equations, using an extended Green's function.

Let $G(P, Q)$ denote a function of the points P and Q on the surface of

* No confusion need arise from the use of s as a suffix as well as to denote length of arc.

the sea, such that

$$\left\{\frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial}{\partial y}\right)\right\} G(P, Q) = 0,$$

except when P coincides with Q, x, y being the coordinates of P and h referring to the position of P. Further, let $G(P, Q)$ be such that

$$\int h\frac{\partial}{\partial n} G(P, Q)\, ds = -1,$$

$\partial/\partial n$ and ds referring to P, and the integral being taken round any circuit enclosing Q; and such that

$$h\frac{\partial}{\partial n} G(P, Q) \to \text{constant},$$

at the boundary, with

$$\int \{G(P, Q)-G(P, Q')\}\, ds = 0,$$

for all positions of Q and Q', the integral being taken round the boundary. Such a function exists uniquely inside the boundary except when P coincides with Q, in which case it has a logarithmic infinity, and it may be shown in the ordinary way* that

$$G(P, Q) = G(Q, P),$$

for all positions of P and Q.

Again, let $G(Q)$ be a function of Q such that

$$\iint \{G(P, Q)+G(Q)\}\, dS = 0,$$

for all positions of Q, dS referring to P. Then it is easily verified that every solution of the integral equation

$$\phi(P) = \lambda \iint \{G(P, Q)+G(Q)\}\, \phi(Q)\, dS, \tag{29}$$

where dS refers to Q, is also a solution of the equations (24) and (25).†

Consider also the integral equation

$$\phi(P) = \lambda \iint \{G(P, Q)+G(P)+G(Q)\}\, \phi(Q)\, dS, \tag{30}$$

* Cf., for instance, Goursat, *Cours d'Analyse*, 2nd ed., t. 3, § 519.

† Cf., for instance, Heywood and Fréchet, *L'Equation de Fredholm*, p. 127; Goursat, *l.c.*, p. 529.

dS again referring to Q. Every solution of (29) is also a solution of (30), and every solution of (30) satisfying (26) is also a solution of (29). Now for a solution of (30) we have

$$\left\{1-\lambda\iint G(P)\,dS\right\}\iint\phi(P)\,dS=0,$$

so that the only solutions of (30) which are not also solutions of (29) are those corresponding to

$$\lambda=\left\{\iint G(P)\,dS\right\}^{-1}=\lambda_0,$$

say.

Let us consider solutions of

$$\phi(P)=\lambda_0\iint\{G(P,Q)+G(P)+G(Q)\}\,\phi(Q)\,dS,$$

dS referring to Q, with the condition

$$\iint\phi(P)\,dS=1.$$

One solution is given by

$$\phi(P)=\text{constant},$$

and the difference between two solutions satisfies (29) with $\lambda=\lambda_0$, from which we deduce that the only solution of (30) which is linearly independent of those of (29) is a constant.

Now suppose that $F(P)$ is a function of P, integrable, together with its square, over the sea, such that

$$\iint F(P)\,dS=0, \tag{31}$$

and take*

$$P(P)=\iint\{G(P,Q)+G(Q)\}F(Q)\,dS=\iint\{G(P,Q)+G(P)+G(Q)\}F(Q)\,dS,$$

where dS refers to Q, so that

$$\left\{\frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial}{\partial y}\right)\right\}P(P)=-F(P), \tag{32}$$

over the sea, with

$$h\frac{\partial}{\partial n}P(P)\to 0, \tag{33}$$

at the boundary, and

$$\iint P(P)\,dS=0. \tag{34}$$

* No confusion need arise from the use of P as the symbol of a function as well as that of a point.

Since the nucleus of the Fredholm's equation (30) is symmetrical, it follows from Hilbert's expansion theorem* that $P(P)$ can be expressed in an absolutely and uniformly convergent series in the solutions of (30), and therefore of (24) and (25), since from (34) there will be no constant term.

Also, the vanishing of $P(P)$ over the sea requires that of $F(P)$, so that the nucleus of the equation (30) is "closed", and this requires the solutions of (30) to form a "complete system",† so that the vanishing of

$$\iint F(P)\,\phi(P)\,dS$$

for all the solutions of (30) requires $F(P)$ to be zero, and for all solutions of (29) requires $F(P)$ to be a constant.

Let all the independent solutions of (24) and (25) be

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_r,\ \ldots,$$

corresponding respectively to the constants

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_r,\ \ldots,$$

arranged in non-decreasing order of magnitude. From (27) and (28) we may take

$$\iint h\left(\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_s}{\partial x}+\frac{\partial\phi_r}{\partial y}\frac{\partial\phi_s}{\partial y}\right)dS=\lambda_r\iint\phi_r\phi_s dS=\lambda_s\iint\phi_r\phi_s dS$$

$$=0\quad(s\neq r),\tag{28}$$

$$=1\quad(s=r).\tag{35}$$

If
$$a_r=\iint F\phi_r dS=\lambda_r\iint P\phi_r dS,\tag{36}$$

for all positive integral values of r, we have, from the expansion theorem quoted,

$$P=\sum_{r=1}^{\infty}a_r\phi_r,\tag{37}$$

the series being absolutely and uniformly convergent.

Take also a function F', which is doubly differentiable over the sea, and

$$a'_r=\iint F'\phi_r dS,\tag{38}$$

* "Grundzüge einen allegemeinen Theorie der linearen Integralgleichungen," *Gött. Nach.*, math. phys. Klasse, Heft 1 (1904). See, for instance, Heywood and Fréchet, *l.c.*, p. 97; Goursat, *l.c.*, p. 447.

† See, for instance, Heywood and Fréchet, *l.c.*, p. 95; Goursat, *l.c.*, p. 443.

for all positive integral values of r. Then

$$\begin{aligned}
\iint FF'dS &= -\iint F'\left\{\frac{\partial}{\partial x}\left(h\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial P}{\partial y}\right)\right\}dS \\
&= -\iint P\left\{\frac{\partial}{\partial x}\left(h\frac{\partial F'}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial F'}{\partial y}\right)\right\}dS+\int hP\frac{\partial F'}{\partial n}ds \\
&= -\sum_{r=1}^{\infty} a_r\left[\iint \phi_r\left\{\frac{\partial}{\partial x}\left(h\frac{\partial F'}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial F'}{\partial y}\right)\right\}dS-\int h\phi_r\frac{\partial F'}{\partial n}ds\right] \\
&= -\sum_{r-1}^{\infty} a_r\iint F'\left\{\frac{\partial}{\partial x}\left(h\frac{\partial \phi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \phi_r}{\partial y}\right)\right\}dS \\
&= \sum_{r=1}^{\infty}\lambda_r a_r\iint F'\phi_r dS \\
&= \sum_{r=1}^{\infty}\lambda_r a_r a'_r, \qquad (39)
\end{aligned}$$

which is Parseval's relation, and which we could have obtained with no more restriction on F' than on F, from a known theorem* by virtue of the completeness of the system of functions ϕ_r. It will be noticed that only one of the functions F, F' need satisfy (31).

Bessel's relation,

$$\iint\left(F-\sum_{r=1}^{N}\lambda_r a_r\phi_r\right)^2 dS = \iint F^2dS-\sum_{r=1}^{N}\lambda_r a_r^2,$$

shows the absolute convergence of (39), and gives, by virtue of (39) with $F'=F$,

$$F = \sum_{r=1}\lambda_r a_r\phi_r, \qquad (40)$$

for any finite area over which the series is convergent.

By the Riesz-Fischer theorem† the absolute convergence of

$$\sum_{=1}^{\infty}\lambda_r a_r^2 \qquad (41)$$

is a sufficient condition for the existence of a function F of the prescribed type satisfying (36).

---

* F. Riesz, "Sur les systèmes orthogonaux de fonctions et l'équation de Fredholm," *Comptes rendus*, t. 144, p. 734 (1907).

† See, for instance, W. H. and G. C. Young, "On the Theorem of Riesz-Fischer", *Quart. Journ. Math.*, Vol. 44, p. 49 (1913); the case of functions defined over an area is given by F. Riesz, *l.c.*, p. 735.

4. Consider also a solution of the differential equation

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial y}\right)+\mu_r\psi_r=0, \tag{42}$$

over the sea, μ_r being a constant, with the condition

$$\psi_r/h \to 0, \tag{43}$$

at the boundary. We have, again using Green's theorem,

$$\begin{aligned}\mu_r\iint\psi_r^2 dS &= -\iint\psi_r\left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial y}\right)\right\}dS \\ &= \iint\frac{1}{h}\left\{\left(\frac{\partial \psi_r}{\partial x}\right)^2+\left(\frac{\partial \psi_r}{\partial y}\right)^2\right\}dS,\end{aligned} \tag{44}$$

showing that μ_r is positive; while, if ψ_r and ψ_s are solutions corresponding to unequal constants μ_r and μ_s, we have

$$\begin{aligned}\mu_r\iint\psi_r\psi_s dS &= -\iint\psi_s\left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial y}\right)\right\}dS \\ &= \iint\frac{1}{h}\left(\frac{\partial \psi_r}{\partial x}\frac{\partial \psi_s}{\partial x}+\frac{\partial \psi_r}{\partial y}\frac{\partial \psi_s}{\partial y}\right)dS \\ &= \mu_s\iint\psi_r\psi_s dS \\ &= 0.\end{aligned} \tag{45}$$

When several linearly independent solutions of (42) and (43) exist corresponding to the same constant μ_r, we may choose them so that (45) is satisfied for each pair.

Let $G'(P, Q)$ be a function of P and Q such that over the sea

$$\left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial}{\partial y}\right)\right\}G'(P, Q)=0,$$

except when P coincides with Q, x, y being the coordinates of P and h referring to P. Further, let $G'(P, Q)$ be such that

$$\int\frac{1}{h}\frac{\partial}{\partial n}G'(P, Q)\,ds=-1,$$

$\partial/\partial n$ and ds referring to P and the integral being taken round any circuit enclosing Q, with

$$G'(P, Q)/h \to 0,$$

as P tends to the boundary. Such a function exists uniquely, except

when P coincides with Q, in which case it has a logarithmic infinity, and again it may be shown in the ordinary way that

$$G'(P, Q) = G'(Q, P),$$

for all positions of P and Q over the sea.

It is easily verified that any solution of the integral equation

$$\psi(P) = \mu \iint G'(P, Q)\psi(Q)\, dS, \tag{46}$$

where dS refers to Q, is also a solution of (42) and (43).

Let all the independent solutions of (42) and (43) be

$$\psi_1, \psi_2, \ldots, \psi_r, \ldots,$$

corresponding respectively to the constants

$$\mu_1, \mu_2, \ldots, \mu_r, \ldots,$$

arranged in non-decreasing order of magnitude. We may take

$$\iint \frac{1}{h}\left(\frac{\partial\psi_r}{\partial x}\frac{\partial\psi_s}{\partial x} + \frac{\partial\psi_r}{\partial y}\frac{\partial\psi_s}{\partial y}\right) dS = \mu_r \iint \psi_r\psi_s\, dS = \mu_s \iint \psi_r\psi_s\, dS$$

$$= 0 \quad (s \neq r), \tag{45}$$

$$= 1 \quad (s = r). \tag{47}$$

If F is a function integrable, together with its square, over the sea, and a function Q is taken such that

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial Q}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial Q}{\partial y}\right) = -F, \tag{48}$$

over the sea, with

$$Q/h \to 0, \tag{49}$$

at the boundary, and

$$b_r = \iint F\psi_r\, dS = \mu_r \iint Q\psi_r\, dS, \tag{50}$$

for all positive integral values of r, we have, again using Hilbert's expansion theorem,

$$Q = \sum_{r=1}^{\infty} b_r\psi_r, \tag{51}$$

the series being absolutely and uniformly convergent.

Again, if F' is another function integrable, together with its square, over the sea, and

$$b'_r = \iint F'\psi_r\, dS, \tag{52}$$

for all positive integral values of r, we have Parseval's relation

$$\iint FF' \, dS = \sum_{r=1}^{\infty} \mu_r b_r b'_r, \tag{53}$$

the series being absolutely convergent.

Again, we have the analogue of (40), while, by the Riesz-Fischer theorem, the absolute convergence of

$$\sum_{r=1}^{\infty} \mu_r b_r^2 \tag{54}$$

is a sufficient condition for the existence of a function F of the prescribed type satisfying (50).

Gyroscopic Coefficients.

5. In the lemma of § 2 take, for all positive integral values of r,

$$X = -\frac{\partial \phi_r}{\partial y}, \quad Y = \frac{\partial \phi_r}{\partial x}, \quad P = \phi'_r, \quad Q = \psi''_r,$$

thus defining ϕ'_r and ψ''_r so that

$$\frac{\partial}{\partial x}\left(h\frac{\partial \phi'_r}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial \phi'_r}{\partial y}\right) = \frac{\partial}{\partial x}\left(h\frac{\partial \phi_r}{\partial y}\right) - \frac{\partial}{\partial y}\left(h\frac{\partial \phi_r}{\partial x}\right), \tag{55}$$

over the sea, with

$$h\frac{\partial \phi'_r}{\partial n} \to h\frac{\partial \phi_r}{\partial s}, \tag{56}$$

at the boundary, and

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi''_r}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi''_r}{\partial y}\right) = \frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}, \tag{57}$$

over the sea, with

$$\psi''_r/h \to 0, \tag{58}$$

at the boundary, while

$$\left.\begin{aligned} -\frac{\partial \phi_r}{\partial y} &= -\frac{\partial \phi'_r}{\partial x} - \frac{1}{h}\frac{\partial \psi''_r}{\partial y}, \\ \frac{\partial \phi_r}{\partial x} &= -\frac{\partial \phi'_r}{\partial y} + \frac{1}{h}\frac{\partial \psi''_r}{\partial x}, \end{aligned}\right\} \tag{59}$$

over the sea.

Again, in the lemma of § 2, take, for all positive integral values of r,

$$X = \frac{1}{h}\frac{\partial \psi_r}{\partial x}, \quad Y = \frac{1}{h}\frac{\partial \psi_r}{\partial y}, \quad P = \phi''_r, \quad Q = \psi'_r,$$

thus defining ϕ_r'' and ψ_r' so that

$$\frac{\partial}{\partial x}\left(h\frac{\partial \phi_r''}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \phi_r''}{\partial y}\right)=-\left(\frac{\partial^2\psi_r}{\partial x^2}+\frac{\partial^2\psi_r}{\partial y^2}\right), \tag{60}$$

over the sea, with

$$h\frac{\partial \phi_r''}{\partial n}\to-\frac{\partial \psi_r}{\partial n}, \tag{61}$$

at the boundary, and

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r'}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r'}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial x}\right), \tag{62}$$

over the sea, with

$$\psi_r'/h\to 0, \tag{63}$$

at the boundary, while

$$\left.\begin{aligned}\frac{1}{h}\frac{\partial \psi_r}{\partial x}&=-\frac{\partial \phi_r''}{\partial x}-\frac{1}{h}\frac{\partial \psi_r'}{\partial y},\\ \frac{1}{h}\frac{\partial \psi_r}{\partial y}&=-\frac{\partial \phi_r''}{\partial y}+\frac{1}{h}\frac{\partial \psi_r'}{\partial x},\end{aligned}\right\} \tag{64}$$

over the sea.

Take now

$$\begin{aligned}\beta_{r,s}&=\lambda_s\iint \phi_r'\phi_s\,dS\\ &=\iint h\left(\frac{\partial \phi_r'}{\partial x}\frac{\partial \phi_s}{\partial x}+\frac{\partial \phi_r'}{\partial y}\frac{\partial \phi_s}{\partial y}\right)dS\\ &=-\iint h\left(\frac{\partial \phi_r}{\partial x}\frac{\partial \phi_s}{\partial y}-\frac{\partial \phi_r}{\partial y}\frac{\partial \phi_s}{\partial x}\right)dS\\ &=\iint \phi_r\left\{\frac{\partial}{\partial x}\left(h\frac{\partial \phi_s}{\partial y}\right)-\frac{\partial}{\partial y}\left(h\frac{\partial \phi_s}{\partial x}\right)\right\}dS-\int h\phi_r\frac{\partial \phi_s}{\partial s}\,ds,\end{aligned} \tag{65}$$

$$\begin{aligned}\beta_{-r,s}&=\lambda_s\iint \phi_r''\phi_s\,dS\\ &=\iint h\left(\frac{\partial \phi_r''}{\partial x}\frac{\partial \phi_s}{\partial x}+\frac{\partial \phi_r''}{\partial y}\frac{\partial \phi_s}{\partial y}\right)dS\\ &=-\iint\left(\frac{\partial \psi_r}{\partial x}\frac{\partial \phi_s}{\partial x}+\frac{\partial \psi_r}{\partial y}\frac{\partial \phi_s}{\partial y}\right)dS,\end{aligned} \tag{66}$$

$$\begin{aligned}\beta_{r,-s}&=\mu_s\iint \psi_r''\psi_s\,dS\\ &=\iint\frac{1}{h}\left(\frac{\partial \psi_r''}{\partial x}\frac{\partial \psi_s}{\partial x}+\frac{\partial \psi_r''}{\partial y}\frac{\partial \psi_s}{\partial y}\right)dS\\ &=\iint\left(\frac{\partial \phi_r}{\partial x}\frac{\partial \psi_s}{\partial x}+\frac{\partial \phi_r}{\partial y}\frac{\partial \psi_s}{\partial y}\right)dS,\end{aligned} \tag{67}$$

and $$\beta_{-r,-s} = \mu_s \iint \psi'_r \psi_s \, dS$$

$$= -\iint \psi_s \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi'_r}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi'_r}{\partial y}\right) \right\} dS$$

$$= -\iint \psi_s \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r}{\partial x}\right) \right\} dS$$

$$= -\iint \frac{1}{h}\left(\frac{\partial \psi_r}{\partial x}\frac{\partial \psi_s}{\partial y} - \frac{\partial \psi_r}{\partial y}\frac{\partial \psi_s}{\partial x}\right) dS, \qquad (68)$$

for all positive integral values of r and s. We notice that

$$\beta_{s,r} = -\beta_{r,s}, \qquad (69)$$

for all integral values of r and s, positive and negative, and we shall see in the next section that these constants play the part of gyroscopic coefficients.

On applying (39) and (53) we have a number of relations of which an example is

$$\sum_{t=1}^{\infty} \beta_{r,-t}\beta_{s,-t} = -\iint \psi''_s \left(\frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}\right) dS, \qquad (70)$$

for all positive integral values of r and s, since

$$\beta_{r,-t} = -\iint \psi_t \left(\frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}\right) dS. \qquad (71)$$

Now

$$\iint h \left\{ \left(\frac{\partial \phi_r}{\partial y} - \sum_{s=1}^{M} \beta_{r,s}\frac{\partial \phi_s}{\partial x} - \frac{1}{h}\sum_{s=1}^{N} \beta_{r,-s}\frac{\partial \psi_s}{\partial y}\right)^2 + \left(\frac{\partial \phi_r}{\partial x} + \sum_{s=1}^{M} \beta_{r,s}\frac{\partial \phi_s}{\partial y} - \frac{1}{h}\sum_{s=1}^{N} \beta_{r,-s}\frac{\partial \psi_s}{\partial x}\right)^2 \right\} dS$$

$$= \iint h \left\{ \left(\frac{\partial \phi_r}{\partial x}\right)^2 + \left(\frac{\partial \phi_r}{\partial y}\right)^2 \right\} dS - \sum_{s=1}^{M} \beta_{r,s}^2 - \sum_{s=1}^{N} \beta_{r,-s}^2,$$

and $$\iint h \left\{ \left(\frac{1}{h}\frac{\partial \psi_r}{\partial x} + \sum_{s=1}^{M} \beta_{-r,s}\frac{\partial \phi_s}{\partial x} + \frac{1}{h}\sum_{s=1}^{N} \beta_{-r,-s}\frac{\partial \psi_s}{\partial y}\right)^2 + \left(\frac{1}{h}\frac{\partial \psi_r}{\partial y} + \sum_{s=1}^{M} \beta_{-r,s}\frac{\partial \phi_s}{\partial y} - \frac{1}{h}\sum_{s=1}^{N} \beta_{-r,-s}\frac{\partial \psi_s}{\partial x}\right)^2 \right\} dS$$

$$= \iint \frac{1}{h} \left\{ \left(\frac{\partial \psi_r}{\partial x}\right)^2 + \left(\frac{\partial \psi_r}{\partial y}\right)^2 \right\} dS - \sum_{s=1}^{M} \beta_{-r,s}^2 - \sum_{s=1}^{N} \beta_{-r,-s}^2,$$

giving, by virtue of (35) and (47),

$$\sum_{s=-\infty}^{\infty} \beta_{r,s}^2 \leqslant 1, \tag{72}$$

for all positive and negative integral values of r. We immediately deduce

$$\sum_{t=-\infty}^{\infty} |\beta_{r,t}\beta_{s,t}| \leqslant 1, \tag{73}$$

for all positive and negative integral values of r and s.

For a *sea of uniform depth* we have, from (65), (67) and (68),

$$\beta_{r,s} = -h\int \phi_r \frac{\partial \phi_s}{\partial s} ds, \tag{65}'$$

$$\beta_{r,-s} = \frac{\lambda_r}{h}\iint \phi_r \psi_s dS, \tag{67}'$$

$$\beta_{-r,-s} = 0, \tag{68}'$$

for all positive integral values of r and s.

Principal Coordinates.

6. Returning now to the subject and notation of § 2, let us take

$$p_r = -\iint \xi\phi_r dS = \lambda_r \iint \phi\phi_r dS, \tag{74}$$

and

$$p_{-r} = -\iint \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\psi_r dS = \mu_r \iint \psi\psi_r dS, \tag{75}$$

for all positive integral values of r.*

From §§ 3, 4, we have

$$\left.\begin{aligned} \phi &= \sum_{r=1}^{\infty} p_r\phi_r, \\ \psi &= \sum_{r=1}^{\infty} p_{-r}\psi_r, \end{aligned}\right\} \tag{76}$$

the series being absolutely and uniformly convergent over the whole sea, while

$$\zeta = -\sum_{r=1}^{\infty} \lambda_r p_r \phi_r, \tag{77}$$

* There is no necessary connection between p_r and p_{-r}.

for any finite area over which this series is convergent, and

$$\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y} = -\sum_{r=1}^{\infty} \mu_r p_{-r} \psi_r, \tag{78}$$

for any finite area over which this series is convergent.

From (4) and (17), we have

$$\frac{2T}{\rho} = \iint h \left\{ \left(\frac{\partial \dot{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \dot{\phi}}{\partial y}\right)^2 \right\} dS + \iint \frac{1}{h} \left\{ \left(\frac{\partial \dot{\psi}}{\partial x}\right)^2 + \left(\frac{\partial \dot{\psi}}{\partial y}\right)^2 \right\} dS,$$

and
$$\begin{aligned}
\iint h \left\{ \left(\frac{\partial \dot{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \dot{\phi}}{\partial y}\right)^2 \right\} dS &= -\iint \dot{\phi} \left\{ \frac{\partial}{\partial x}\left(h \frac{\partial \dot{\phi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial \dot{\phi}}{\partial y}\right) \right\} dS \\
&= -\sum_{r=1}^{\infty} \dot{p}_r \iint \phi_r \left\{ \frac{\partial}{\partial x}\left(h \frac{\partial \dot{\phi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial \dot{\phi}}{\partial y}\right) \right\} dS \\
&= \sum_{r=1}^{\infty} \lambda_r \dot{p}_r \iint \dot{\phi} \phi_r \, dS \\
&= \sum_{r=1}^{\infty} \dot{p}_r^2,
\end{aligned}$$

while

$$\begin{aligned}
&\iint \frac{1}{h} \left\{ \left(\frac{\partial \dot{\psi}}{\partial x}\right)^2 + \left(\frac{\partial \dot{\psi}}{\partial y}\right)^2 \right\} dS \\
&\quad = -\iint \dot{\psi} \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h} \frac{\partial \dot{\psi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h} \frac{\partial \dot{\psi}}{\partial y}\right) \right\} dS \\
&\quad = -\sum_{r=1}^{\infty} \dot{p}_{-r} \iint \psi_r \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h} \frac{\partial \dot{\psi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h} \frac{\partial \dot{\psi}}{\partial y}\right) \right\} dS \\
&\quad = \sum_{r=1}^{\infty} \mu_r \dot{p}_{-r} \iint \dot{\psi} \psi_r \, dS \\
&\quad = \sum_{r=1}^{\infty} \dot{p}_{-r}^2,
\end{aligned}$$

so that
$$\frac{2T}{\rho} = \sum_{r=-\infty}^{\infty} \dot{p}_r^2, \tag{79}$$

the series converging absolutely. Also from (5) and (39) we have

$$\frac{2V}{\rho} = \sum_{r=1}^{\infty} \nu_r^2 p_r^2, \tag{80}$$

where
$$\nu_r^2 = g\lambda_r, \tag{81}$$

for all positive integral values of r, the series again converging absolutely; similarly

$$\iint \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)^2 dS = \sum_{r=1}^{\infty} \mu_r p_{-r}^2, \tag{82}$$

the series converging absolutely.

Again,

$$\lambda_r \iint \phi' \phi_r dS = \iint h \left(\frac{\partial \phi'}{\partial x}\frac{\partial \phi_r}{\partial x} + \frac{\partial \phi'}{\partial y}\frac{\partial \phi_r}{\partial y}\right) dS$$

$$= -\iint h \left(\frac{\partial \phi}{\partial x}\frac{\partial \phi_r}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \phi_r}{\partial x}\right) dS$$

$$-\iint \left(\frac{\partial \psi}{\partial x}\frac{\partial \phi_r}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \phi_r}{\partial y}\right) dS,$$

while $$\iint h \left(\frac{\partial \phi}{\partial x}\frac{\partial \phi_r}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \phi_r}{\partial x}\right) dS = \iint h \left(\frac{\partial \phi}{\partial x}\frac{\partial \phi_r'}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \phi_r'}{\partial y}\right) dS$$

$$= -\iint \phi_r' \xi dS$$

$$= \sum_{s=1}^{\infty} \beta_{r,s} p_s,$$

from (39), and

$$\iint \left(\frac{\partial \psi}{\partial x}\frac{\partial \phi_r}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \phi_r}{\partial y}\right) dS = -\iint \psi \left(\frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}\right) dS$$

$$= -\iint \psi \left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi_r''}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi_r''}{\partial y}\right)\right\} dS$$

$$= -\iint \psi_r'' \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right) dS$$

$$= \sum_{s=1}^{\infty} \beta_{r,-s} p_{-s},$$

from (53), so that $$\lambda_r \iint \phi' \phi_r dS = -\sum_{s=-\infty}^{\infty} \beta_{r,s} p_s, \tag{83}$$

the series being absolutely convergent.

Further,

$$\mu_r \iint \psi' \psi_r \, dS = \iint \frac{1}{h} \left(\frac{\partial \psi'}{\partial x} \frac{\partial \psi_r}{\partial x} + \frac{\partial \psi'}{\partial y} \frac{\partial \psi_r}{\partial y} \right) dS$$

$$= \iint \left(\frac{\partial \phi}{\partial x} \frac{\partial \psi_r}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi_r}{\partial y} \right) dS$$

$$- \iint \frac{1}{h} \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi_r}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \psi_r}{\partial x} \right) dS,$$

while $$\iint \left(\frac{\partial \phi}{\partial x} \frac{\partial \psi_r}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi_r}{\partial y} \right) dS = - \iint h \left(\frac{\partial \phi}{\partial x} \frac{\partial \phi_r''}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi_r''}{\partial y} \right) dS$$

$$= \iint \phi_r'' \xi \, dS$$

$$= - \sum_{s=1}^{\infty} \beta_{-r,s} p_s,$$

from (39), and

$$\iint \frac{1}{h} \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi_r}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \psi_r}{\partial x} \right) dS = \iint \frac{1}{h} \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi_r'}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi_r'}{\partial y} \right) dS$$

$$= - \iint \psi_r' \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y} \right) dS$$

$$= \sum_{s=1}^{\infty} \beta_{-r,-s} p_{-s},$$

from (53), so that $$\mu_r \iint \psi' \psi_r \, dS = - \sum_{s=-\infty}^{\infty} \beta_{-r,s} p_s, \qquad (84)$$

the series being absolutely convergent.

From (6), (17), and (22), we have

$$\frac{M}{\rho} = - \iint h \left(\frac{\partial \phi}{\partial x} \frac{\partial \phi'}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \phi'}{\partial y} \right) dS - \iint \frac{1}{h} \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi'}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi'}{\partial y} \right) dS$$

$$= \iint \phi' \dot{\xi} \, dS + \iint \psi' \left(\frac{\partial \dot{\eta}}{\partial x} - \frac{\partial \dot{\xi}}{\partial y} \right) dS$$

$$= \sum_{r=1}^{\infty} \left(\sum_{s=-\infty}^{\infty} \beta_{r,s} p_s \right) \dot{p}_r + \sum_{r=1}^{\infty} \left(\sum_{s=-\infty}^{\infty} \beta_{-r,s} p_s \right) \dot{p}_{-r}$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \beta_{r,s} \dot{p}_r p_s, \qquad (85)^*$$

* In the notation of Part I the functions M and B are now identical.

on using (39) and (53), the repeated series necessarily converging. Also,

$$\begin{aligned}\frac{M}{\rho} &= \iint h\left(\frac{\partial\phi}{\partial x}\frac{\partial\dot{\phi}'}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\dot{\phi}'}{\partial y}\right)dS+\iint\frac{1}{h}\left(\frac{\partial\psi}{\partial x}\frac{\partial\dot{\psi}'}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\dot{\psi}'}{\partial y}\right)dS\\ &= -\iint\dot{\phi}'\xi\,dS-\iint\dot{\psi}'\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right)dS\\ &= -\sum_{r=1}^{\infty}\left(\sum_{s=-\infty}^{\infty}\beta_{r,s}\dot{p}_s\right)p_r-\sum_{r=1}^{\infty}\left(\sum_{s=-\infty}^{\infty}\beta_{-r,s}\dot{p}_s\right)p_{-r}\\ &= \sum_{s=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\beta_{r,s}\dot{p}_r p_s, \qquad (86)^*\end{aligned}$$

again using (39) and (53), the repeated series again necessarily converging.

Multiply the first equation in (23) by $\lambda_r\phi_r$, the second by $\mu_r\psi_r$, and integrate over the whole area of the sea; we obtain respectively

$$\left.\begin{aligned}\ddot{p}_r+2\omega\sum_{s=-\infty}^{\infty}\beta_{r,s}\dot{p}_s+\nu_r^2 p_r &= P_r,\\ \ddot{p}_{-r}+2\omega\sum_{s=-\infty}^{\infty}\beta_{-r,s}\dot{p}_s &= 0,\end{aligned}\right\} \qquad (87)$$

where

$$P_r = \lambda_r\iint\Omega\phi_r\,dS, \qquad (88)$$

for all positive integral values of r, by virtue of (74), (75), (83), and (84).

From the way in which we have obtained the equations (87), it follows that they have a solution corresponding to every solution of the equations (23). We shall now proceed to show that every solution of the equations (87), which makes the series in (80) and (82) absolutely convergent, gives a solution of the equations of § 2.

It follows from §§ 3, 4 that, when the series in (80) and (82) are absolutely convergent, functions

$$\xi, \quad \frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y},$$

exist which satisfy (74) and (75). With these take functions ϕ and ψ in accordance with (13)–(16); these will give ξ, η by (17), and then ϕ' and ψ' can be taken in accordance with (18)–(22). The functions ϕ and ψ so obtained will satisfy (74) and (75), while the functions ϕ' and ψ' so ob-

* See Note on previous page.

tained will satisfy (83) and (84). From the completeness of the systems of functions ϕ_r and ψ_r, equations (87) for all positive integral values of r require (23), and therefore we have a solution of the equations of § 2.*

The equations (87) *may therefore be taken as the equations of motion in the general tidal problem.* We notice from (65)–(68), that they may be explicitly constructed when all the functions ϕ_r and ψ_r are known.

We see from (79), (80), (85), and (86), that the equations (87) may be written in the form

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{p}_r} - \frac{\partial R}{\partial p_r} = P_r, \tag{89}$$

for all integral values of r, positive and negative, where

$$\rho R = T + \omega M - V, \tag{90}$$

and

$$P_{-r} = 0,$$

for positive values of r.

We have thus obtained a set of equations of the same type as those discussed in Part I, the set of parameters p_r, for all positive and negative integral values of r, playing the part of *principal coordinates.*

7. For a *sea of uniform depth* we see from (68)′ that p_{-r}, for positive values of r, are coordinates of the *third class*, and that there are no coordinates of the second class. The sea forms a system of Type II, and there are no oscillations of the second class. The elimination of the coordinates of the third class in periodic motion can be effected exactly as in § 4 of Part I, and we obtain

$$-\sigma^2 p_r + 2\omega i\sigma \sum_{s=1}^{\infty} \beta_{r,s} p_s + \nu_r^2 p_r + 4\omega^2 \frac{\partial}{\partial p_r} K\{p\} = P_r, \tag{91}$$

for all positive integral values of r, where

$$2K\{p\} = \sum_{r=1}^{\infty} \left(\sum_{s=1}^{\infty} \beta_{-r,s} p_s \right)^2, \tag{92}$$

which can easily be shown convergent by our usual methods. The coefficient of p_s in $\partial K\{p\}/\partial p_r$ is

$$\sum_{t=1}^{\infty} \beta_{r,-t} \beta_{s,-t},$$

the value of which is given by (70).

* The Riesz-Fischer theorem does not guarantee the continuity of ζ and $\partial\eta/\partial x - \partial\xi/\partial y$, and a discontinuous ζ cannot be interpreted physically, but this does not affect the proof that the equations (87) are equivalent to those of § 2.

For a *slowly rotating sea*, the methods of Lord Rayleigh, developed in § 6 of Part I, are directly applicable to the equations (89). For instance, if no other coefficient of stability is equal to ν_r^2, the corresponding free speed is given by

$$\sigma = \pm \nu_r \left(1 + 2\omega^2 \sum_{s=-\infty}^{\infty} \frac{\beta_{r,s}^2}{\nu_r^2 - \nu_s^2}\right), \tag{93}$$

to the second order in ω, the series having no term for which $s = r$, and being absolutely convergent from (72), while

$$p_s = \frac{2\omega i\sigma}{\nu_r^2 - \nu_s^2} \beta_{s,r} p_r \tag{94}$$

if $s \neq r$.

To investigate *coordinates of the second class*, we are led by §§ 3, 4 of Part I to consider the system whose equations are

$$\ddot{p}_{-r} + 2\omega \sum_{s=1}^{\infty} \beta_{-r,-s} \dot{p}_{-s} = 0, \tag{95}$$

for all positive integral values of r. These are obtained from (87) by suppressing all the coordinates (74); let us therefore consider a solution of the second equation in (23) under the condition $\phi = 0$. For this purpose we have

$$\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi'}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi}{\partial x}\right),$$

and

$$\psi' = i\beta\psi,$$

in periodic motion of speed $2\omega\beta$, and therefore

$$i\beta\left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi}{\partial y}\right)\right\} - \left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial \psi}{\partial x}\right)\right\} = 0. \tag{96}$$

Suppose that for a certain value of β, $\psi = E$ is a solution of (96) over the sea, with

$$E/h \to 0, \tag{97}$$

at the boundary. The function E will be complex, and when the sign of β is changed in (96), E will be changed to its conjugate E'.

Take $$a_{-r} = \mu_r \iint E\psi_r dS, \qquad a_{-r'} = \mu_r \iint E'\psi_r dS, \tag{98}$$

for all positive integral values of r. We have then

$$a_{-r} = -\iint \psi_r \left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial E}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial E}{\partial y}\right)\right\} dS, \tag{99}$$

and from (53), (68) and (98),

$$\sum_{s=1}^{\infty} \beta_{-r,\,-s}\,\alpha_{-s} = -\iint E \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\,\frac{\partial \psi_r}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{h}\,\frac{\partial \psi_r}{\partial x}\right)\right\} dS$$

$$= \iint \psi_r \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\,\frac{\partial E}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{h}\,\frac{\partial E}{\partial x}\right)\right\} dS. \qquad (100)$$

Multiply (96) by ψ_r and integrate over the area of the sea; we obtain from (99) and (100),

$$-i\beta\alpha_{-r} - \sum_{s=1}^{\infty} \beta_{-r,\,-s}\,\alpha_{-s} = 0, \qquad (101)$$

for all positive integral values of r, which shows that a solution of (95) is obtained by taking

$$p_{-r} = \alpha_{-r} e^{i\sigma t}, \qquad \sigma = 2\omega\beta.$$

Now take

$$f = \sum_{r=1}^{\infty} \alpha_{-r}\, p_{-r} = -\iint \psi \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\,\frac{\partial E'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\,\frac{\partial E'}{\partial y}\right)\right\} dS, \qquad (102)$$

$$\beta'_r = -\sum_{s=1}^{\infty} \beta_{r,\,-s}\,\alpha_{-s} = \iint E' \left(\frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}\right) dS. \qquad (103)$$

We have

$$\ddot{f} - 2\omega i\beta \dot{f} = -\iint (\ddot{\psi} - 2\omega i\beta\dot{\psi}) \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\,\frac{\partial E'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\,\frac{\partial E'}{\partial y}\right)\right\} dS$$

$$= -2\omega \iint \dot{\psi}' \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\,\frac{\partial E'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\,\frac{\partial E'}{\partial y}\right)\right\} dS$$

$$-2\omega \iint \dot{\psi} \left\{ \frac{\partial}{\partial x}\left(\frac{1}{h}\,\frac{\partial E'}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{h}\,\frac{\partial E'}{\partial x}\right)\right\} dS$$

$$= 2\omega \iint \frac{1}{h} \left\{ \frac{\partial E'}{\partial x}\left(\frac{\partial \dot{\psi}'}{\partial x} - \frac{\partial \dot{\psi}}{\partial y}\right) + \frac{\partial E'}{\partial y}\left(\frac{\partial \dot{\psi}'}{\partial y} + \frac{\partial \dot{\psi}}{\partial x}\right)\right\} dS$$

$$= 2\omega \iint \left(\frac{\partial E'}{\partial x}\,\frac{\partial \dot{\phi}}{\partial x} + \frac{\partial E'}{\partial y}\,\frac{\partial \dot{\phi}}{\partial y}\right) dS$$

$$= -2\omega \iint \dot{\phi}\left(\frac{\partial^2 E'}{\partial x^2} + \frac{\partial^2 E'}{\partial y^2}\right) dS + 2\omega \int \dot{\phi}\,\frac{\partial E'}{\partial n}\, ds$$

$$= -2\omega \sum_{r=1}^{\infty} \dot{p}_r \left\{ \iint \phi_r \left(\frac{\partial^2 E'}{\partial x^2} + \frac{\partial^2 E'}{\partial y^2}\right) dS - \int \phi_r \frac{\partial E'}{\partial n}\, ds \right\}$$

$$= -2\omega \sum_{r=1}^{\infty} \dot{p}_r \iint E' \left(\frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}\right) dS$$

$$= -2\omega \sum_{r=1}^{\infty} \beta'_r \dot{p}_r,$$

so that $$\ddot{f} - 2\omega i \beta \dot{f} + 2\omega \sum_{r=1}^{\infty} \beta'_r \dot{p}_r = 0, \tag{104}$$

showing that f is a coordinate of the second class.

Examples.

8. Let us now consider a sea in which the depth is a function only of the distance r from a fixed point, the boundaries being circles having this point for centre.

It is convenient to replace by double suffixes the single suffixes in ϕ_r, ψ_r, as in the present examples these functions are naturally arranged in double sequences.

Using polar coordinates r, θ, we have

$$\left.\begin{aligned} \phi_{2s,\,n} &= \phi_s^{(n)}(r) \cos n\theta, \\ \phi_{2s+1,\,n} &= \phi_s^{(n)}(r) \sin n\theta, \end{aligned}\right\} \tag{105}*$$

for all positive integral values of n and s, where

$$\left\{ \frac{1}{r} \frac{d}{dr}\left(rh \frac{d}{dr}\right) - \frac{n^2 h}{r^2} + \lambda_s^{(n)} \right\} \phi_s^{(n)}(r) = 0, \tag{106}$$

over the range of r, and

$$h \frac{d}{dr} \phi_s^{(n)}(r) \to 0, \tag{107}$$

at the bounds, with $$\pi \lambda_s^{(n)} \int \{\phi_s^{(n)}(r)\}^2 r\, dr = 1, \tag{108}$$

the integration and those that follow being over the range of r. We have also

$$\phi_{s,\,0} = \phi_s^{(0)}(r), \tag{105}'$$

and the equations (106) and (107) with $n = 0$, while

$$2\pi \lambda_s^{(0)} \int \{\phi_s^{(0)}(r)\}^2 r\, dr = 1. \tag{108}'$$

Again, we have

$$\left.\begin{aligned} \psi_{2s,\,n} &= \psi_s^{(n)}(r) \cos n\theta, \\ \psi_{2s+1,\,n} &= \psi_s^{(n)}(r) \sin n\theta, \end{aligned}\right\} \tag{109}*$$

* No confusion need arise from the use of r as a coordinate as well as a suffix.

for all positive integral values of n and s, where

$$\left\{\frac{1}{r}\frac{d}{dr}\left(\frac{r}{h}\frac{d}{dr}\right)-\frac{n^2}{r^2h}+\mu_s^{(n)}\right\}\psi_s^{(n)}(r)=0, \tag{110}$$

over the range of r, and $$\psi_s^{(n)}(r)/h\to 0, \tag{111}$$

at the bounds, with $$\pi\mu_s^{(n)}\int\{\psi_s^{(n)}(r)\}^2 r\,dr=1. \tag{112}$$

We have also $$\psi_{s,0}=\psi_s^{(0)}(r), \tag{109)'}$$

and the equations (110) and (111) with $n=0$, while

$$2\pi\mu_s^{(0)}\int\{\psi_s^{(0)}(r)\}^2 r\,dr=1. \tag{112)'}$$

We see at once from (65)–(68) that all the gyroscopic coefficients which are derived from functions with different values of n will vanish. From this it follows that all the coordinates corresponding to one value of n form a *fundamental class.*

Let us first consider the fundamental class corresponding to $n=0$, in which the elevation of the free surface ζ will be a function only of r. We notice that all the gyroscopic coefficients vanish except those given by (66) and (67), so that we have no coordinates of the second class. The fundamental class forms a system of *Type II*, and the coordinates of the third class may be eliminated from the equations of motion.

We easily obtain, from (70),

$$\sum_{t=1}^{\infty}\beta_{r,-t}\beta_{s,-t}=0 \quad (s\neq r), \tag{113}$$

$$=1 \quad (s=r), \tag{114}$$

for positive integral values of r and s, so that the equations of motion reduce to $$(-\sigma^2+\nu_r^2+4\omega^2)p_r=P_r, \tag{115}$$

for all positive integral values of r. We therefore have, for the free speeds,

$$\sigma=\pm(\nu_r^2+4\omega^2)^{\frac{1}{2}}, \tag{116}$$

the corresponding modes being given by

$$p_s=0, \tag{117}$$

for positive values of s other than r. Under these circumstances we have, from (77),

$$\zeta=-\lambda_r p_r\phi^{(0)}(r). \tag{118}$$

For the fundamental class corresponding to a *general value of* n, we immediately see that

$$\beta^{(n)}_{2r,\,2s} = \beta^{(n)}_{2r+1,\,2s+1} = \beta^{(n)}_{2r+1,\,-2s} = \beta^{(n)}_{2r,\,-2s-1} = \beta^{(n)}_{-2r,\,-2s} = \beta^{(n)}_{-2r-1,\,-2s-} = 0, \tag{119}$$

for all positive integral values of r and s.

Let us now take the case of *uniform depth*, the sea being circular and of radius a. We have

$$\phi^{(n)}_s(r) = \left(\frac{2}{\pi h}\right)^{\frac{1}{2}} \frac{J_n(\kappa_s r)}{\{(\kappa_s a)^2 - n^2\}^{\frac{1}{2}} J_n(\kappa_s a)}, \tag{120}$$

with
$$J'_n(\kappa_s a) = 0, \quad \lambda^{(n)}_s = h\kappa_s^2, \tag{121}$$

and
$$\psi^{(n)}_s(r) = \left(\frac{2h}{\pi}\right)^{\frac{1}{2}} \frac{J_n(k_s r)}{k_s a J'_n(k_s a)}, \tag{122}$$

with
$$J_n(k_s a) = 0, \quad \mu^{(n)}_s = k_s^2/h. \tag{123}$$

We have also
$$\beta_{2r,\,2s+1} = -\beta_{2r+1,\,2s}$$
$$= -\frac{2n}{\{(\kappa_r a)^2 - n^2\}^{\frac{1}{2}} \{(\kappa_s a)^2 - n^2\}^{\frac{1}{2}}}, \tag{124}$$
$$\beta_{2r,\,-2s} = \beta_{2r+1,\,-2s-1}$$
$$= \frac{2\kappa_r^2}{(\kappa_r^2 - k_s^2)\{(\kappa_r a)^2 - n^2\}^{\frac{1}{2}}}. \tag{125}$$

The equations of motion become

$$\left.\begin{aligned} \ddot{p}_{2r} + 2\omega \sum_{s=1}^{\infty} \beta_{2r,\,2s+1} \dot{p}_{2s+1} + \nu_r^2 p_{2r} \quad + 2\omega \sum_{s=1}^{\infty} \beta_{2r,\,-2s} \dot{p}_{-2s} \quad &= P_{2r}, \\ \ddot{p}_{2r+1} - 2\omega \sum_{s=1}^{\infty} \beta_{2r,\,2s+1} \dot{p}_{2s} \quad + \nu_r^2 p_{2r+1} + 2\omega \sum_{s=1}^{\infty} \beta_{2r,\,-2s} \dot{p}_{-2s-1} &= P_{2r+1}, \end{aligned}\right\} \tag{126}$$

and
$$\left.\begin{aligned} \ddot{p}_{-2r} \quad + 2\omega \sum_{s=1}^{\infty} \beta_{-2r,\,2s} \dot{p}_{2s} \quad &= 0, \\ \ddot{p}_{-2r-1} + 2\omega \sum_{s=1}^{\infty} \beta_{-2r,\,2s} \dot{p}_{2s+1} &= 0, \end{aligned}\right\} \tag{127}$$

for all positive integral values of r. On taking

$$\sqrt{2}\, f_r = p_{2r} + i p_{2r+1}, \qquad \sqrt{2}\, f_r = p_{2r} - i p_{2r+1}, \tag{128}$$

$$\sqrt{2}\, F_r = P_{2r} + i P_{2r+1}, \quad \sqrt{2}\, F_r = P_{2r} - i P_{2r+1}, \tag{129}$$

we obtain, for the equations of motion,

$$\ddot{f}_r+2\omega i\sum_{s=1}^{\infty}\beta_{2r,\,2s+1}\dot{f}_s+\nu_r^2 f_r+2\omega\sum_{s=1}^{\infty}\beta_{2r,\,-2s}\dot{f}_{-s}=F_r, \tag{130}$$

$$\ddot{f}_{-r}+2\omega\sum_{s=1}^{\infty}\beta_{-2r,\,2s}\dot{f}_s=0, \tag{131}$$

for all positive integral values of r, with a similar set of equations involving only the coordinates $f_{r'}$ and $f_{-r'}$. We therefore see that the set of coordinates f_r and f_{-r}, for all positive integral values of r, forms of itself a fundamental class, while the set of coordinates $f_{r'}$ and $f_{-r'}$ forms another fundamental class. In the corresponding modes of oscillation ζ will contain θ only in the factors $e^{in\theta}$ and $e^{-in\theta}$ respectively.

For the elimination of the coordinates of the third class, we easily obtain

$$\sum_{t=1}^{\infty}\beta_{2r,\,-2t}\beta_{2s,\,-2t}=\beta_{2r,\,2s+1}\quad(s\neq r), \tag{132}$$

$$=1+\beta_{2r,\,2r+1}\quad(s=r), \tag{133}$$

for all positive integral values of r and s. The equations (87) then yield

$$(\nu_r^2-\sigma^2+4\omega^2)f_r-2\omega(\sigma-2\omega)\sum_{s=1}^{\infty}\beta_{2r,\,2s+1}f_s=F_r,$$

which may be written

$$\left(1-\frac{\sigma^2-4\omega^2}{\nu_r^2}\right)\nu_r f_r-2\omega(\sigma-2\omega)\sum_{s=1}^{\infty}\frac{\beta_{2r,\,2s+1}}{\nu_r\nu_s}\nu_s f_s=\frac{F_r}{\nu_r}, \tag{134}$$

for all positive integral values of r.

Since $$\sum_{s=1}^{\infty}\frac{\beta_{2r,\,2s+1}^2}{\nu_r^2\nu_s^2}<\frac{1}{\nu_r\nu_1^2},$$

and it may be shown that $\sum_{r=1}^{\infty}\frac{1}{\nu_r^2}$ is convergent,* we see that the equations (134) may be solved by absolutely converging infinite determinants.†

* Prof. J. McMahon's formulæ for $\kappa_r a$ are given by Gray and Mathews, *Treatise on Bessel Functions*, p. 241.

† This follows from a theorem of Von Koch; see, for instance, F. Riesz, *Les Systèmes d'Equations Linéares à une Infinité d'Inconnues*, c. 2 (Paris, 1913).

The determinant whose vanishing gives the free speeds is

$$\begin{vmatrix} 1-\frac{\sigma^2-4\omega^2}{\nu_1^2}+\frac{4n\omega(\sigma-2\omega)}{\nu_1^2\{(\kappa_1 a)^2-n^2\}}, & \frac{4n\omega(\sigma-2\omega)}{\nu_1\nu_2\{(\kappa_1 a)^2-n^2\}^{\frac{1}{2}}\{(\kappa_2 a)^2-n^2\}^{\frac{1}{2}}}, \ldots, & \frac{4n\omega(\sigma-2\omega)}{\nu_1\nu_r\{(\kappa_1 a)^2-n^2\}^{\frac{1}{2}}\{(\kappa_r a)^2-n^2\}^{\frac{1}{2}}}, \ldots \\ \frac{4n\omega(\sigma-2\omega)}{\nu_2\nu_1\{(\kappa_2 a)^2-n^2\}^{\frac{1}{2}}\{(\kappa_1 a)^2-n^2\}^{\frac{1}{2}}}, & 1-\frac{\sigma^2-4\omega^2}{\nu_2^2}+\frac{4n\omega(\sigma-2\omega)}{\nu_2^2\{(\kappa_2 a)^2-n^2\}}, \ldots, & \frac{4n\omega(\sigma-2\omega)}{\nu_2\nu_r\{(\kappa_2 a)^2-n^2\}^{\frac{1}{2}}\{(\kappa_r a)^2-n^2\}^{\frac{1}{2}}}, \ldots \\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \\ \frac{4n\omega(\sigma-2\omega)}{\nu_r\nu_1\{(\kappa_r a)^2-n^2\}^{\frac{1}{2}}\{(\kappa_1 a)^2-n^2\}^{\frac{1}{2}}}, & \frac{4n\omega(\sigma-2\omega)}{\nu_r\nu_2\{(\kappa_r a)^2-n^2\}^{\frac{1}{2}}\{(\kappa_2 a)^2-n^2\}^{\frac{1}{2}}}, \ldots, & 1-\frac{\sigma^2-4\omega^2}{\nu_r^2}+\frac{4n\omega(\sigma-2\omega)}{\nu_r^2\{(\kappa_r a)^2-n^2\}}, \ldots \\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \end{vmatrix}$$

and is easily seen to reduce to

$$\prod_{r=1}^{\infty}\left(1-\frac{\sigma^2-4\omega^2}{\nu_r^2}\right)\left[1+\sum_{r=1}^{\infty}\frac{4n\omega(\sigma-2\omega)}{\{(\kappa_r a)^2-n^2\}\{\nu_r^2-\sigma^2+4\omega^2\}}\right].$$

On writing

$$\kappa^2=\frac{\sigma^2-4\omega^2}{gh}, \tag{135}$$

this becomes

$$\prod_{r=1}^{\infty}\left\{1-\frac{(\kappa a)^2}{(\kappa_r a)^2}\right\}\left[1+\frac{4n\omega(\kappa a)^2}{\sigma+2\omega}\sum_{r=1}^{\infty}\frac{1}{\{(\kappa_r a)^2-n^2\}\{(\kappa_r a)^2-(\kappa a)^2\}}\right]. \tag{136}$$

If we write for a moment

$$P(z)=\prod_{r=1}^{\infty}\left\{1-\frac{z}{(\kappa_r a)^2}\right\},$$

we obtain

$$\frac{1}{n^2-(\kappa a)^2}\left[\frac{P'\{(\kappa a)^2\}}{P\{(\kappa a)^2\}}-\frac{P'(n^2)}{P(n^2)}\right]=\sum_{r=1}^{\infty}\frac{1}{\{(\kappa_r a)^2-n^2\}\{(\kappa_r a)^2-(\kappa a)^2\}};$$

but

$$P(z)=\frac{1}{2^n n!}z^{\frac{1}{2}(1-n)}J_n'(z^{\frac{1}{2}}),$$

so that

$$\frac{P'(z)}{P(z)}=-\frac{n}{2z}+\frac{n^2-z}{2z^{\frac{3}{2}}}\frac{J_n(z^{\frac{1}{2}})}{J_n'(z^{\frac{1}{2}})},$$

from which we see that (136) becomes

$$\frac{1}{2^n n!\,(\kappa a)^n(\sigma+2\omega)}\{\sigma\kappa a J_n'(\kappa a)+2n\omega J_n(\kappa a)\}, \tag{137}$$

the vanishing of which gives the known result.*

* Lamb, *l.c.*, 3rd ed., p. 306; 4th ed., p. 312.

Again, in the above determinant, it is easily seen that the co-factor of the element in the r-th row and s-th column is

$$\frac{4n\omega(\sigma-2\omega)\prod_{r=1}^{\infty}\left(1-\frac{\sigma^2-4\omega^2}{\nu_r^2}\right)}{\nu_r\nu_s\left\{(\kappa_r a)^2-n^2\right\}^{\frac{1}{2}}\left\{(\kappa_s a)^2-n^2\right\}^{\frac{1}{2}}\left(1-\frac{\sigma^2-4\omega^2}{\nu_r^2}\right)\left(1-\frac{\sigma^2-4\omega^2}{\nu_s^2}\right)}, \quad (138)$$

showing that the free mode is given by

$$\nu_r f_r = \frac{ghC}{\nu_r\left\{(\kappa_r a)^2-n^2\right\}^{\frac{1}{2}}\left(1-\frac{\sigma^2-4\omega^2}{\nu_r^2}\right)},$$

or

$$f_r = \frac{C}{\left\{(\kappa_r a)^2-n^2\right\}^{\frac{1}{2}}(\kappa_r^2-\kappa^2)}, \quad (139)$$

where C is a constant. This also is in accordance with the known result*

$$\xi = AJ_n(\kappa_r)e^{in\theta+i\sigma t},$$

where A is a constant, since

$$\int_0^a J_n(\kappa r)\,\phi_r^{(n)}(r)\,r\,dr = \left(\frac{2}{\pi h}\right)^{\frac{1}{2}}\frac{\kappa a J_n'(\kappa a)}{\left\{(\kappa_r a)^2-n^2\right\}^{\frac{1}{2}}(\kappa_r^2-\kappa^2)}.$$

As an example of *oscillations of the second class* let us take the case in which

$$h = h_0\left(1-\frac{r^2}{a^2}\right), \quad (140)$$

r ranging from 0 to a. With θ entering only through the factor $e^{in\theta}$, the equation (96) becomes

$$r^2(a^2-r^2)\frac{\partial^2 E}{\partial r^2}+r(a^2+r^2)\frac{\partial E}{\partial r}-\left\{n^2(a^2-r^2)+\frac{2n}{\beta}r^2\right\}E=0, \quad (141)$$

giving

$$E = Ae^{in\theta}\left(\frac{r}{a}\right)\ F\left\{\frac{n-1-\alpha}{2},\ \frac{n-1+\alpha}{2},\ n+1,\ \left(\frac{r}{a}\right)^2\right\}, \quad (142)$$

in the ordinary notation of hypergeometric functions, where A is a constant, and

$$\alpha^2 = n^2+1-\frac{2n}{\beta}. \quad (143)$$

* Lamb, *l.c.*, 3rd ed., p. 306; 4th ed., p. 312.

On $r = a$, (142) becomes

$$\frac{A e^{in\theta}\,\Pi(n)}{\Pi\left(\frac{n+1+\alpha}{2}\right)\Pi\left(\frac{n+1-\alpha}{2}\right)},$$

in Gauss's notation, which will vanish if $\frac{1}{2}(n+1-\alpha)$ is a negative integer, making $\partial E/\partial r = 0$ on $r = a$, so that (97) will be satisfied. On writing $\sigma = 2\omega\beta$, we obtain results agreeing with Lamb.*

* *L.c.*, 3rd ed., p. 311 ; 4th ed., p. 317.

ON THE DYNAMICAL EQUATIONS OF THE TIDES

Part III.—Oceans on a Sphere.

By J. Proudman.

[Read March 9th, 1916.]

1. In this part the results obtained in Part II for flat rotating seas are extended to oceans on a rotating sphere.

Again, the algebraic linear equations which result from the transformation can be constructed in a number of cases for which the tidal problem has not yet been solved. But in the present part their solution is only discussed in cases where the depth is a function of the latitude only, the boundaries, if any, being parallels of latitude. This class includes all the cases for which the tidal problem has hitherto been completely solved; for each there exists an infinite set of fundamental classes of coordinates.

In stating the analytical basis of the transformation, only such equations as are afterwards quoted, or would be required in applications, are given. When the reasoning is exactly the same as in Part II, it is omitted. The numbering of sections and equations is throughout the same as that of corresponding sections and equations of Part II.

General Differential Equations.

2. Let the radius of the sphere be denoted by a, and, using the axis of rotation as axis of spherical polar coordinates, let θ denote the co-latitude of any point on the sphere, and χ the longitude of the point measured from some meridian fixed on the sphere.

We suppose the ocean to satisfy the analytical conditions stated in § 2 of Part II.

Let h denote the depth at any point below the free surface in equilibrium, and l, m the cosines of the angles which any outward drawn normal to the boundary makes with the directions respectively of θ, χ increasing.

The remarks at the beginning of § 2 of Part II concerning differentia-

tion, boundary conditions, surface integrals and line integrals, again apply.

Let ξ, η denote the components, in the directions respectively of θ, χ increasing, of the horizontal displacement of a particle of water, and ζ the elevation of the free surface at the point whose coordinates are θ, χ. Then ξ, η, ζ will be functions only of θ, χ and the time t, and the equation of continuity gives

$$\frac{1}{a \sin\theta}\left\{\frac{\partial}{\partial\theta}(h \sin\theta\xi)+\frac{\partial}{\partial\chi}(h\eta)\right\}+\zeta=0, \tag{1}$$

over the ocean. We have also

$$h(l\xi+m\eta)\to 0, \tag{2}$$

at the boundary, which with (1) gives

$$\iint \zeta\, dS=0. \tag{3}$$

The kinetic energy T of the relative motion is given by

$$\frac{2T}{\rho}=\iint h(\dot{\xi}^2+\dot{\eta}^2)\, dS, \tag{4}$$

where ρ denotes the density of the water, supposed uniform, and the potential energy V by

$$\frac{2V}{\rho}=g\iint \zeta^2\, dS, \tag{5}$$

where g denotes the acceleration due to gravity, supposed uniform. We have here neglected the mutual gravitation of the elevated water. Also the bilinear part M of the moment of momentum of the relative motion about the axis is given by

$$\frac{M}{\rho}=\iint h\cos\theta\,(\xi\dot{\eta}-\eta\dot{\xi})\, dS. \tag{6}$$

If the sphere is rotating with constant angular velocity ω, and if Ω denotes the potential of the disturbing forces, the dynamical equations of motion* are

$$\left.\begin{aligned}\ddot{\xi}-2\omega\cos\theta\dot{\eta}&=-\frac{1}{a}\frac{\partial}{\partial\theta}(g\zeta+\Omega),\\ \ddot{\eta}+2\omega\cos\theta\dot{\xi}&=-\frac{1}{a\sin\theta}\frac{\partial}{\partial\chi}(g\zeta+\Omega),\end{aligned}\right\} \tag{7}$$

to be satisfied over the whole ocean.

* See, for instance, Lamb, *Hydrodynamics*, 3rd ed., p. 314; 4th ed., p. 320.

Let X, Y be two functions which are differentiable over the ocean. Take a function P so that

$$\frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial P}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial P}{\partial\chi}\right)\right\}=-\left\{\frac{\partial}{\partial\theta}(h\sin\theta X)+\frac{\partial}{\partial\chi}(hY)\right\} \quad (8)$$

over the ocean, with

$$h\frac{\partial P}{\partial n}\to -h(lX+mY) \quad (9)$$

at the boundary; this determines P except to an additive constant. Take also a function Q such that

$$\frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial Q}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial Q}{\partial\chi}\right)\right\}=\frac{\partial}{\partial\theta}(\sin\theta Y)-\frac{\partial X}{\partial\chi} \quad (10)$$

over the sea, with

$$Q/h\to 0 \quad (11)$$

at the boundary. Then exactly as in Part II it may be proved that

$$\left.\begin{aligned} X&=-\frac{1}{a}\frac{\partial P}{\partial\theta}-\frac{1}{ha\sin\theta}\frac{\partial Q}{\partial\chi},\\ Y&=-\frac{1}{a\sin\theta}\frac{\partial P}{\partial\chi}+\frac{1}{ha}\frac{\partial Q}{\partial\theta},\end{aligned}\right\} \quad (12)$$

over the ocean. Of this lemma we shall make considerable use.

Take $X=\xi,\quad Y=\eta,\quad P=\phi,\quad Q=\psi,$

so that we have

$$\frac{1}{a^2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial\phi}{\partial\chi}\right)\right\}=\zeta \quad (13)$$

over the ocean, with

$$h\frac{\partial\phi}{\partial n}\to 0 \quad (14)$$

at the boundary,

$$\frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\psi}{\partial\chi}\right)\right\}=\frac{\partial}{\partial\theta}(\sin\theta\eta)-\frac{\partial\xi}{\partial\chi} \quad (15)$$

over the ocean, with

$$\psi/h\to 0 \quad (16)$$

at the boundary, and

$$\left.\begin{aligned} \xi&=-\frac{1}{a}\frac{\partial\phi}{\partial\theta}-\frac{1}{ha\sin\theta}\frac{\partial\psi}{\partial\chi},\\ \eta&=-\frac{1}{a\sin\theta}\frac{\partial\phi}{\partial\chi}+\frac{1}{ha}\frac{\partial\psi}{\partial\theta},\end{aligned}\right\} \quad (17)$$

over the ocean.

Again, in the lemma, take

$$X = \cos\theta\eta, \quad Y = -\cos\theta\xi, \quad P = \phi', \quad Q = \psi',$$

so that we have

$$\frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial\phi'}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial\phi'}{\partial\chi}\right)\right\}$$

$$= -\left\{\frac{\partial}{\partial\theta}(h\sin\theta\cos\theta\eta)-\frac{\partial}{\partial\chi}(h\cos\theta\xi)\right\} \quad (18)$$

over the ocean, with

$$h\frac{\partial\phi'}{\partial n} \to -h\cos\theta(l\eta - m\xi) \quad (19)$$

at the boundary,

$$\frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\psi'}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\psi'}{\partial\chi}\right)\right\}$$

$$= -\left\{\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\xi)+\frac{\partial}{\partial\chi}(\cos\theta\eta)\right\} \quad (20)$$

over the ocean, with

$$\psi'/h \to 0 \quad (21)$$

at the boundary, and

$$\left.\begin{aligned}\cos\theta\eta &= -\frac{1}{a}\frac{\partial\phi'}{\partial\theta}-\frac{1}{ha\sin\theta}\frac{\partial\psi'}{\partial\chi},\\ -\cos\theta\xi &= -\frac{1}{a\sin\theta}\frac{\partial\phi'}{\partial\chi}+\frac{1}{ha}\frac{\partial\psi'}{\partial\theta},\end{aligned}\right\} \quad (22)$$

over the ocean.

Then it is easily seen that equations (7) are equivalent to

$$\left.\begin{aligned}\ddot{\phi}-2\omega\dot{\phi}'-(g\zeta+\Omega) &= \text{constant},\\ \ddot{\psi}-2\omega\dot{\psi}' &= 0,\end{aligned}\right\} \quad (23)$$

so that we may take these as the equations of the tidal problem instead of (2) and (7).

Analytical Basis of Transformation.

3. Consider solutions of the differential equation

$$\frac{1}{a^2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial\phi_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial\phi_r}{\partial\chi}\right)\right\}+\lambda_r\phi_r = 0. \quad (24)$$

over the ocean, λ_r being a constant, with the condition

$$h\frac{\partial \phi_r}{\partial n} \to 0 \qquad (25)$$

at the boundary. Then exactly as in Part II it may be shown that there exists an infinite sequence of linearly independent solutions

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_r,\ \ldots,$$

corresponding respectively to the positive constants

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_r,\ \ldots,$$

arranged in non-decreasing order of magnitude, satisfying the relations

$$\frac{1}{a^2}\iint h\left(\frac{\partial \phi_r}{\partial \theta}\frac{\partial \phi_s}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial \phi_r}{\partial \chi}\frac{\partial \phi_s}{\partial \chi}\right)dS$$

$$= \lambda_r \iint \phi_r \phi_s\, dS = \lambda_s \iint \phi_r \phi_s\, dS$$

$$= 0 \quad (s \neq r), \qquad (28)$$

$$= 1 \quad (s = r), \qquad (35)$$

and possessing the property of "completeness" when a constant is added to the sequence.

If F is a function of position, integrable, together with its square, over the ocean, such that

$$\iint F\,dS = 0, \qquad (31)$$

and a function P is taken so that

$$\frac{1}{a^2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial P}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial P}{\partial\chi}\right)\right\} = -F \qquad (32)$$

over the ocean, with

$$h\frac{\partial P}{\partial n} \to 0 \qquad (33)$$

at the boundary, and

$$a_r = \iint F\phi_r\,dS = \lambda_r \iint P\phi_r\,dS, \qquad (36)$$

for all positive integral values of r, we have

$$P = \sum_{r=1}^{\infty} a_r \phi_r, \qquad (37)$$

the series being absolutely and uniformly convergent.

If F' is another function integrable, together with its square, over the ocean, and

$$a'_r = \iint F' \phi_r dS, \tag{38}$$

for all positive integral values of r, we have Parseval's relation

$$\iint FF' dS = \sum_{r=1}^{\infty} \lambda_r a_r a'_r, \tag{39}$$

the series converging absolutely.

We have also

$$F = \sum_{r=1}^{\infty} \lambda_r a_r \phi_r, \tag{40}$$

for any finite area over which this series is convergent.

By the Riesz-Fischer theorem the absolute convergence of

$$\sum_{r=1}^{\infty} \lambda_r a_r^2 \tag{41}$$

is a sufficient condition for the existence of a function F of the prescribed type satisfying (36).

4. Consider also solutions of the differential equation

$$\frac{1}{a^2 \sin\theta} \left\{ \frac{\partial}{\partial\theta} \left(\frac{\sin\theta}{h} \frac{\partial \psi_r}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\chi} \left(\frac{1}{h} \frac{\partial \psi_r}{\partial\chi} \right) \right\} + \mu_r \psi_r = 0 \tag{42}$$

over the ocean, with the condition

$$\psi_r / h \to 0 \tag{43}$$

at the boundary. Then again it may be shown that there exists an infinite sequence of linearly independent solutions

$$\psi_1, \psi_2, \ldots, \psi_r, \ldots$$

corresponding respectively to the positive constants

$$\mu_1, \mu_2, \ldots, \mu_r, \ldots,$$

arranged in non-decreasing order of magnitude, satisfying the relations

$$\begin{aligned} \frac{1}{a^2} \iint \frac{1}{h} \left(\frac{\partial \psi_r}{\partial\theta} \frac{\partial \psi_s}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial \psi_r}{\partial\chi} \frac{\partial \psi_s}{\partial\chi} \right) dS & \\ = \mu_r \iint \psi_r \psi_s dS = \mu_s \iint \psi_r \psi_s dS & \\ = 0 \quad (s \neq r), & \tag{45} \\ = 1 \quad (s = r), & \tag{47} \end{aligned}$$

and possessing the property of "completeness".

If F is a function of position integrable, together with its square, over the ocean, and a function Q is taken so that

$$\frac{1}{a^2 \sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial Q}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial Q}{\partial\chi}\right)\right\}=-F \tag{48}$$

over the ocean, with

$$Q/h \to 0 \tag{49}$$

at the boundary, and

$$b_r = \iint F\psi_r dS = \mu_r \iint Q\psi_r dS, \tag{50}$$

for all positive integral values of r, we have

$$Q = \sum_{r=1}^{\infty} b_r \psi_r, \tag{51}$$

the series being absolutely and uniformly convergent.

Again, if F' is another function integrable, together with its square, over the ocean, and

$$b'_r = \iint F'\psi_r dS, \tag{52}$$

for all positive integral values of r, we have Parseval's relation

$$\iint FF' dS = \sum_{r=1}^{\infty} \mu_r b_r b'_r, \tag{53}$$

the series being absolutely convergent.

Again, we have the analogue of (40), while by the Riesz-Fischer theorem, the absolute convergence of

$$\sum_{r=1}^{\infty} \mu_r b_r^2 \tag{54}$$

is a sufficient condition for the existence of a function F of the prescribed type satisfying (50).

Gyroscopic Coefficients.

5. In the lemma of § 2 take, for all positive integral values of r,

$$X = -\frac{\cos\theta}{a\sin\theta}\frac{\partial\phi_r}{\partial\chi}, \qquad Y = \frac{\cos\theta}{a}\frac{\partial\phi_r}{\partial\theta}, \qquad P = \phi'_r, \qquad Q = \psi''_r,$$

thus defining ϕ'_r and ψ''_r, so that

$$\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial\phi'_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial\phi'_r}{\partial\chi}\right)=\frac{\partial}{\partial\theta}\left(h\cos\theta\frac{\partial\phi_r}{\partial\chi}\right)-\frac{\partial}{\partial\chi}\left(h\cos\theta\frac{\partial\phi_r}{\partial\theta}\right) \tag{55}$$

over the ocean, with

$$h\frac{\partial\phi_r'}{\partial n}\to h\cos\theta\frac{\partial\phi_r}{\partial s} \tag{56}$$

at the boundary, and

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\psi_r''}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\psi_r''}{\partial\chi}\right)$$

$$=\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\frac{\partial\phi_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\cos\theta\frac{\partial\phi_r}{\partial\chi}\right) \tag{57}$$

over the ocean, with

$$\psi_r''/h\to 0 \tag{58}$$

at the boundary, while

$$\left.\begin{aligned}-\frac{\cos\theta}{\sin\theta}\frac{\partial\phi_r}{\partial\chi}&=-\frac{\partial\phi_r'}{\partial\theta}-\frac{1}{h\sin\theta}\frac{\partial\psi_r''}{\partial\chi},\\ \cos\theta\frac{\partial\phi_r}{\partial\theta}&=-\frac{1}{\sin\theta}\frac{\partial\phi_r'}{\partial\chi}+\frac{1}{h}\frac{\partial\psi_r''}{\partial\theta},\end{aligned}\right\} \tag{59}$$

over the ocean.

Again, in the lemma of § 2 take, for all positive integral values of r,

$$X=\frac{\cos\theta}{ha}\frac{\partial\psi_r}{\partial\theta},\qquad Y=\frac{\cos\theta}{ha\sin\theta}\frac{\partial\psi_r}{\partial\chi},\qquad P=\phi_r'',\qquad Q=\psi_r',$$

thus defining ϕ_r'' and ψ_r' so that

$$\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial\phi_r''}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial\phi_r''}{\partial\chi}\right)$$

$$=-\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\frac{\partial\psi_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\cos\theta\frac{\partial\psi_r}{\partial\chi}\right)\right\} \tag{60}$$

over the ocean, with

$$h\frac{\partial\phi_r''}{\partial n}\to-\cos\theta\frac{\partial\psi_r}{\partial n} \tag{61}$$

at the boundary, and

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\psi_r'}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\psi_r'}{\partial\chi}\right)=\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{h}\frac{\partial\psi_r}{\partial\chi}\right)-\frac{\partial}{\partial\chi}\left(\frac{\cos\theta}{h}\frac{\partial\psi_r}{\partial\theta}\right)$$

(62)

over the ocean, with

$$\psi_r'/h\to 0 \tag{63}$$

at the boundary, while

$$\left.\begin{aligned} \frac{\cos\theta}{h}\frac{\partial\psi_r}{\partial\theta} &= -\frac{\partial\phi_r''}{\partial\theta} - \frac{1}{h\sin\theta}\frac{\partial\psi_r'}{\partial\chi}, \\ \frac{\cos\theta}{h\sin\theta}\frac{\partial\psi_r}{\partial\chi} &= -\frac{1}{\sin\theta}\frac{\partial\phi_r''}{\partial\chi} + \frac{1}{h}\frac{\partial\psi_r'}{\partial\theta}, \end{aligned}\right\} \tag{64}$$

over the ocean.

Take now

$$\begin{aligned} \beta_{r,s} &= \lambda_s \iint \phi_r' \phi_s \, dS \\ &= \frac{1}{a^2}\iint h\left(\frac{\partial\phi_r'}{\partial\theta}\frac{\partial\phi_s}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\phi_r'}{\partial\chi}\frac{\partial\phi_s}{\partial\chi}\right) dS \\ &= -\frac{1}{a^2}\iint \frac{h\cos\theta}{\sin\theta}\left(\frac{\partial\phi_r}{\partial\theta}\frac{\partial\phi_s}{\partial\chi} - \frac{\partial\phi_r}{\partial\chi}\frac{\partial\phi_s}{\partial\theta}\right) dS \\ &= \frac{1}{a^2}\iint \frac{\phi_r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(h\cos\theta\frac{\partial\phi_s}{\partial\chi}\right) - \frac{\partial}{\partial\chi}\left(h\cos\theta\frac{\partial\phi_s}{\partial\theta}\right)\right\} dS \\ &\qquad - \int h\cos\theta\phi_r\frac{\partial\phi_s}{\partial s}\, ds, \end{aligned} \tag{65}$$

$$\begin{aligned} \beta_{-r,s} &= \lambda_s \iint \phi_r'' \phi_s \, dS \\ &= \frac{1}{a^2}\iint h\left(\frac{\partial\phi_r''}{\partial\theta}\frac{\partial\phi_s}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\phi_r''}{\partial\chi}\frac{\partial\phi_s}{\partial\chi}\right) dS \\ &= -\frac{1}{a^2}\iint \cos\theta\left(\frac{\partial\psi_r}{\partial\theta}\frac{\partial\phi_s}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\psi_r}{\partial\chi}\frac{\partial\phi_s}{\partial\chi}\right) dS, \end{aligned} \tag{66}$$

$$\begin{aligned} \beta_{r,-s} &= \mu_s \iint \psi_r'' \psi_s \, dS \\ &= \frac{1}{a^2}\iint \frac{1}{h}\left(\frac{\partial\psi_r''}{\partial\theta}\frac{\partial\psi_s}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\psi_r''}{\partial\chi}\frac{\partial\psi_s}{\partial\chi}\right) dS \\ &= \frac{1}{a^2}\iint \cos\theta\left(\frac{\partial\phi_r}{\partial\theta}\frac{\partial\psi_s}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\phi_r}{\partial\chi}\frac{\partial\psi_s}{\partial\chi}\right) dS, \end{aligned} \tag{67}$$

and

$$\beta_{-r,-s} = \mu_s \iint \psi'_r \psi_s \, dS$$

$$= -\frac{1}{a^2} \iint \frac{\psi_s}{\sin\theta} \left\{ \frac{\partial}{\partial\theta} \left(\frac{\sin\theta}{h} \frac{\partial \psi'_r}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\chi} \left(\frac{1}{h} \frac{\partial \psi'_r}{\partial\chi} \right) \right\} dS$$

$$= -\frac{1}{a^2} \iint \frac{\psi_s}{\sin\theta} \left\{ \frac{\partial}{\partial\theta} \left(\frac{\cos\theta}{h} \frac{\partial \psi_r}{\partial\chi} \right) - \frac{\partial}{\partial\chi} \left(\frac{\cos\theta}{h} \frac{\partial \psi_r}{\partial\theta} \right) \right\} dS$$

$$= -\frac{1}{a^2} \iint \frac{\cos\theta}{h \sin\theta} \left(\frac{\partial \psi_r}{\partial\theta} \frac{\partial \psi_s}{\partial\chi} - \frac{\partial \psi_r}{\partial\chi} \frac{\partial \psi_s}{\partial\theta} \right) dS, \tag{68}$$

for all positive integral values of r and s. We notice that

$$\beta_{s,r} = -\beta_{r,s}, \tag{69}$$

for all integral values of r and s, positive and negative, and we shall see in the next section that these constants play the part of gyroscopic coefficients.

On applying (39) and (53), we have a number of relations of which an example is

$$\sum_{t=1}^{\infty} \beta_{r,-t} \beta_{s,-t} = -\frac{1}{a^2} \iint \frac{\psi''_s}{\sin\theta} \left\{ \frac{\partial}{\partial\theta} \left(\sin\theta \cos\theta \frac{\partial \phi_r}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\chi} \left(\cos\theta \frac{\partial \phi_r}{\partial\chi} \right) \right\} dS, \tag{70}$$

for all positive integral values of r and s, since

$$\beta_{r,-t} = -\frac{1}{a^2} \iint \frac{\psi_t}{\sin\theta} \left\{ \frac{\partial}{\partial\theta} \left(\sin\theta \cos\theta \frac{\partial \phi_r}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\chi} \left(\cos\theta \frac{\partial \phi_r}{\partial\chi} \right) \right\} dS. \tag{71}$$

Exactly as in Part II we may show that

$$\sum_{t=-\infty}^{\infty} |\beta_{r,t} \beta_{s,t}| < 1, \tag{73}$$

for all positive and negative integral values of r and s.

For an ocean over which $h \sec\theta$ *is constant* we have, from (68),

$$\beta_{-r,-s} = 0, \tag{68}'$$

for all positive integral values of r and s.

Principal Coordinates.

6. Returning now to the subject and notation of § 2, let us take

$$p_r = -\iint \zeta \phi_r dS = \lambda_r \iint \phi \phi_r dS, \tag{74}$$

and $$p_{-r} = -\frac{1}{a}\iint \frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\eta) - \frac{\partial\xi}{\partial\chi}\right\}\psi_r dS = \mu_r \iint \psi\psi_r dS, \tag{75}$$

for all positive integral values of r.

From §§ 3, 4 we have

$$\left.\begin{aligned} \phi &= \sum_{r=1}^{\infty} p_r \phi_r, \\ \psi &= \sum_{r=1}^{\infty} p_{-r} \psi_r, \end{aligned}\right\} \tag{76}$$

the series being absolutely and uniformly convergent over the whole ocean, while

$$\zeta = -\sum_{r=1}^{\infty} \lambda_r p_r \phi_r, \tag{77}$$

for any finite area over which this series is convergent, and

$$\frac{1}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\eta) - \frac{\partial\xi}{\partial\chi}\right\} = -\sum_{r=1}^{\infty} \mu_r p_{-r} \psi_r, \tag{78}$$

for any finite area over which this series is convergent.

Exactly as in Part II, we may prove that

$$\frac{2T}{\rho} = \sum_{r=-\infty}^{\infty} \dot{p}_r^2, \tag{79}$$

$$\frac{2V}{\rho} = \sum_{r=1}^{\infty} \nu_r^2 p_r^2, \tag{80}$$

where $$\nu_r^2 = g\lambda_r, \tag{81}$$

for all positive integral values of r, and

$$\frac{1}{a^2}\iint \frac{1}{\sin^2\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta\eta) - \frac{\partial\xi}{\partial\chi}\right\}^2 dS = \sum_{r=1}^{\infty} \mu_r p_{-r}^2, \tag{82}$$

these series converging absolutely.

Again we may show that

$$\lambda_r \iint \phi' \phi_r dS = -\sum_{s=-\infty}^{\infty} \beta_{r,s} p_s, \tag{83}$$

$$\mu_r \iint \psi' \psi_r dS = -\sum_{s=-\infty}^{\infty} \beta_{-r,s} p_s, \tag{84}$$

and

$$\frac{M}{\rho} = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \beta_{r,s} \dot{p}_r p_s, \tag{85}$$

$$= \sum_{s=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \beta_{r,s} \dot{p}_r p_s, \tag{86}$$

the series (83) and (84) being absolutely convergent, and the series (85) and (86) convergent.

Multiply the first equation in (23) by $\lambda_r \phi_r$, the second by $\mu_r \psi_r$ and integrate over the whole area of the ocean; we obtain respectively

$$\left.\begin{aligned} \ddot{p}_r + 2\omega \sum_{s=-\infty}^{\infty} \beta_{r,s} \dot{p}_s + \nu_r^2 p_r &= P_r, \\ \ddot{p}_{-r} + 2\omega \sum_{s=-\infty}^{\infty} \beta_{-r,s} \dot{p}_s &= 0, \end{aligned}\right\} \tag{87}$$

where

$$P_r = \lambda_r \iint \Omega \phi_r dS, \tag{88}$$

for all positive integral values of r, by virtue of (74), (75), (83) and (84).

From the way in which we have obtained the equations (87) it follows that they have a solution corresponding to every solution of the equations (23). We may also show, exactly as in Part II, that every solution of the equations (87) which makes the series in (80) and (82) absolutely convergent, gives a solution of the equation of § 2.

The equations (87) *may therefore be taken as the equations of motion in the general tidal problem.* We notice from (65)–(68) that they may be explicitly constructed when all the functions of ϕ_r and ψ_r are known.

We see from (79), (80), (85) and (86), that the equations (87) may be written in the form

$$\frac{d}{dt} \frac{\partial R}{\partial \dot{p}_r} - \frac{\partial R}{\partial p_r} = P_r, \tag{89}$$

for all integral values of r, positive and negative, where

$$\rho R = T + \omega M - V, \tag{90}$$

and $$P_{-r} = 0,$$

for positive values of r.

We have thus obtained a set of equations of the same type as those discussed in Part I, the set of parameters p_r, for all positive and negative integral values of r, playing the part of *principal coordinates*.

7. For an ocean over which $h \sec\theta$ *is constant*, we see from (68)′ that p_{-r}, for positive values of r, are coordinates of the *third class*, and that there are no coordinates of the second class. The ocean forms a system of *Type II*, and there are no oscillations of the second class. The elimination of the coordinates of the third class in periodic motion can be effected exactly as in § 4 of Part I, and the equations (91) and (92) of Part II again apply.

For a *slowly rotating ocean* the remarks and equations of Part II again apply.

To investigate *coordinates of the second class*, we are led to consider solutions of the second equation in (23) under the condition $\phi = 0$. For this purpose we have

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\psi'}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\psi'}{\partial\chi}\right)=\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{h}\frac{\partial\psi}{\partial\chi}\right)-\frac{\partial}{\partial\chi}\left(\frac{\cos\theta}{h}\frac{\partial\psi}{\partial\theta}\right),$$

and $$\psi' = i\beta\psi,$$

in periodic motion of speed $2\omega\beta$, and therefore

$$i\beta\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\psi}{\partial\chi}\right)\right\}$$
$$-\left\{\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{h}\frac{\partial\psi}{\partial\chi}\right)-\frac{\partial}{\partial\chi}\left(\frac{\cos\theta}{h}\frac{\partial\psi}{\partial\theta}\right)\right\}=0. \quad (96)$$

Suppose that for a certain value of β, $\psi = E$ is a solution of (96) over the ocean, with

$$E/h \to 0 \quad (97)$$

at the boundary. The function E will be complex, and when the sign of β is changed in (96) E will be changed to its conjugate E'.

Take $$a_{-r} = \mu_r \iint E\psi_r\, dS, \quad a_{-r'} = \mu_r \iint E'\psi_r\, dS, \quad (98)$$

for all positive integral values of r. We have then

$$a_{-r} = -\frac{1}{a^2}\iint\frac{\psi_r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial E}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial E}{\partial\chi}\right)\right\}dS, \quad (99)$$

and from (53), (68) and (98),

$$\sum_{s=1}^{\infty} \beta_{-r,s}\alpha_{-s} = -\frac{1}{a^2}\iint \frac{E}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{h}\frac{\partial\psi_r}{\partial\chi}\right)-\frac{\partial}{\partial\chi}\left(\frac{\cos\theta}{h}\frac{\partial\psi_r}{\partial\theta}\right)\right\}dS$$

$$= \frac{1}{a^2}\iint \frac{\psi_r}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{h}\frac{\partial E}{\partial\chi}\right)-\frac{\partial}{\partial\chi}\left(\frac{\cos\theta}{h}\frac{\partial E}{\partial\theta}\right)\right\}dS. \quad (100)$$

Multiply (96) by ψ_r and integrate over the area of the ocean; we obtain from (99) and (100),

$$-i\beta a_{-r} - \sum_{s=1} \beta_{-r,-s}\, a_{-s} = 0, \quad (101)$$

for all positive integral values of r.

Now take

$$f = \sum_{r=1}^{\infty} a_{-r} p_{-r}$$

$$= -\frac{1}{a^2}\iint \frac{\psi}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial E'}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial E'}{\partial\chi}\right)\right\}dS, \quad (102)$$

and

$$\beta'_r = -\sum_{s=1}^{\infty} \beta_{r,-s}\alpha_{-s'}$$

$$= \frac{1}{a^2}\iint \frac{E'}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\frac{\partial\phi_r}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\cos\theta\frac{\partial\phi_r}{\partial\chi}\right)\right\}dS. \quad (103)$$

We can prove, exactly as in Part II, that

$$\ddot{f} - 2\omega i\beta\dot{f} + 2\omega\sum_{r=1}^{\infty}\beta'_r\dot{p} = 0, \quad (104)$$

showing that f is a coordinate of the second class.

Examples.

8. Let us now consider an ocean in which the depth is a function only of latitude, the boundaries, if any, being parallels of latitude. It is convenient to replace by double suffixes the single suffixes in ϕ_r, ψ_r, as in the present examples these functions are naturally arranged in double sequences.

We have

$$\left.\begin{aligned}\phi_{2r,n} &= \phi_r^{(n)}(\theta)\cos n\chi,\\ \phi_{2r+1,n} &= \phi_r^{(n)}(\theta)\sin n\chi,\end{aligned}\right\} \quad (105)$$

for all positive integral values of n and r, where

$$\left\{\frac{1}{a^2\sin\theta}\frac{d}{d\theta}\left(h\sin\theta\frac{d}{d\theta}\right)-\frac{n^2h}{a^2\sin^2\theta}+\lambda_r^{(n)}\right\}\phi_r^{(n)}(\theta)=0 \qquad (106)$$

over the range of θ, and

$$h\frac{d}{d\theta}\phi_r^{(n)}(\theta)\to 0 \qquad (107)$$

at the bounds, with

$$\pi a^2\lambda_r^{(n)}\int\{\phi_r^{(n)}(\theta)\}^2\sin\theta\,d\theta=1, \qquad (108)$$

the integration and those that follow being over the range of θ. We have also

$$\phi_{r,0}=\phi_r^{(0)}(\theta), \qquad (105)'$$

and the equations (106) and (107) with $n=0$, while

$$2\pi a^2\lambda_r^{(0)}\int\{\phi_r^{(0)}(\theta)\}^2\sin\theta\,d\theta=1. \qquad (108)'$$

Again, we have

$$\left.\begin{aligned}\psi_{2r,n}&=\psi_r^{(n)}(\theta)\cos n\chi,\\ \psi_{2r+1,n}&=\psi_r^{(n)}(\theta)\sin n\chi,\end{aligned}\right\} \qquad (109)$$

for all positive integral values of n and r, where

$$\left\{\frac{1}{a^2\sin\theta}\frac{d}{d\theta}\left(\frac{\sin\theta}{h}\frac{d}{d\theta}\right)-\frac{n^2}{a^2h\sin^2\theta}+\mu_r^{(n)}\right\}\psi_r^{(n)}(\theta)=0 \qquad (110)$$

over the range of θ, and

$$\psi_r^{(n)}(\theta)/h\to 0 \qquad (111)$$

at the bounds, with

$$\pi a^2\mu_r^{(n)}\int\{\psi_r^{(n)}(\theta)\}^2\sin\theta\,d\theta=1. \qquad (112)$$

We have also

$$\psi_{r,0}=\psi_r^{(0)}(\theta), \qquad (109)'$$

and the equations (110) and (111) with $n=0$, while

$$2\pi a^2\mu_r^{(0)}\int\{\psi_r^{(0)}(\theta)\}^2\sin\theta\,d\theta=1. \qquad (112)'$$

We see at once from (65)–(68) that all the gyroscopic coefficients which are derived from functions with different values of n will vanish. From this it follows that all the coordinates corresponding to one value of n form a *fundamental class*.

Let us first consider the fundamental class corresponding to $n=0$, in which the elevation of the free surface ζ will be a function of θ only. We notice that all the gyroscopic coefficients vanish except those given by (66)

and (67), so that we have no coordinates of the second class. The fundamental class forms a system of *Type II*, and the coordinates of the third class may be eliminated, with the equations

$$(\nu_r^2-\sigma^2)\, p_r+4\omega^2 \frac{\partial}{\partial p_r} K\{p\} = P_r, \tag{115}$$

for all positive integral values of r, as the result. The coefficient of p_s in $\partial K\{p\}/\partial p_r$ is

$$\sum_{t=1}^{\infty} \beta_{r,\,-t}\beta_{s,\,-t} = 2\pi \int h \frac{d}{d\theta}\phi_r^{(0)}(\theta) \frac{d}{d\theta}\phi_s^{(0)}(\theta) \cos^2\theta \sin\theta\, d\theta, \tag{113}$$

for all positive integral values of r and s, from (70).

For the fundamental class corresponding to a *general value of n*, we immediately see that

$$\beta^{(n)}_{2r,\,2s} = \beta^{(n)}_{2r+1,\,2s+1} = \beta^{(n)}_{2r+1,\,-2s} = \beta^{(n)}_{2r,\,-2s-1} = \beta^{(n)}_{-2r,\,-2s} = \beta^{(n)}_{-2r-1,\,-2s-1} = 0, \tag{119}$$

for all positive integral values of r and s.

Let us now take the case of *uniform depth*, the ocean covering the *whole sphere*. We have

$$\phi_r^{(n)}(\theta) = \left\{ \frac{(2n+2r-1)}{(n+r)(n+r-1)} \frac{(r-1)!}{(2n+r-1)!} \frac{1}{2\pi h} \right\}^{\frac{1}{2}} P^{(n)}_{n+r-1}(\cos\theta), \tag{120}$$

with $$\lambda_r^{(n)} = (n+r)(n+r-1)\, h/a^2, \tag{121}$$

and $$\psi_r^{(n)}(\theta) = \left\{ \frac{(2n+2r-1)}{(n+r)(n+r-1)} \frac{(r-1)!}{(2n+r-1)!} \frac{h}{2\pi} \right\}^{\frac{1}{2}} P^{(n)}_{n+r-1}(\cos\theta), \tag{122}$$

with $$\mu_r^{(n)} = (n+r)(n+r-1)/ha^2. \tag{123}$$

We have also $$\beta^{(n)}_{2r,\,2s+1} = 0,$$

unless $s = r$, while $$\beta^{(n)}_{2r,\,2r+1} = -\frac{n}{(n+r)(n+r-1)}, \tag{124}$$

and $$\beta^{(n)}_{2r,\,-2s} = \beta^{(n)}_{2r+1,\,-2s-1} = 0,$$

unless $r \sim s = 1$, while

$$\beta^{(n)}_{2r,\,-2r+2} = \beta^{(n)}_{2r-2,\,-2r} = \beta^{(n)}_{2r+1,\,-2r+1} = \beta^{(n)}_{2r-1,\,-2r-1}$$

$$= \frac{1}{n+r-1} \left\{ \frac{(r-1)(n+r)(2n+r-1)(2n+r-2)}{(2n+2r-1)(2n+2r-3)} \right\}^{\frac{1}{2}}. \tag{125}$$

Further $$\beta_{-2r,\,-2s-1} = 0,$$

unless $s = r$, while $$\beta_{-2r,\,-2r-1} = -\frac{n}{(n+r)(n+r-1)}. \tag{126}$$

The equations of motion become

$$\left.\begin{aligned} \ddot{p}_{2r} + 2\omega(\beta_{2r,\,2r+1}\dot{p}_{2r+1} + \beta_{2r,\,-2r+2}\dot{p}_{-2r+2} + \beta_{2r,\,-2r-2}\dot{p}_{-2r-2}) \\ + \nu_r^2 p_{2r} = P_{2r}, \\ \ddot{p}_{2r+1} + 2\omega(\beta_{2r+1,\,2r}\dot{p}_{2r} + \beta_{2r+1,\,-2r+1}\dot{p}_{-2r+1} + \beta_{2r+1,\,-2r-3}\dot{p}_{-2r-3}) \\ + \nu_r^2 p_{2r+1} = P_{2r+1}, \end{aligned}\right\} \tag{127}$$

and

$$\left.\begin{aligned} \ddot{p}_{-2r} + 2\omega(\beta_{-2r,\,-2r-1}\dot{p}_{-2r-1} + \beta_{-2r,\,2r-2}\dot{p}_{2r-2} + \beta_{-2r,\,2r+2}\dot{p}_{2r+2}) &= 0, \\ \ddot{p}_{-2r-1} + 2\omega(\beta_{-2r-1,\,-2r}\dot{p}_{-2r} + \beta_{-2r-1,\,2r-1}\dot{p}_{2r-1} + \beta_{-2r-1,\,2r+3}\dot{p}_{2r+3}) &= 0, \end{aligned}\right\} \tag{128}$$

for all positive integral values of r. On taking

$$\sqrt{2}\,f_{r'} = p_{2r} + ip_{2r+1}, \qquad \sqrt{2}\,f_r = p_{2r} - ip_{2r+1}, \tag{129}$$

$$\sqrt{2}\,F_r = P_{2r} + iP_{2r+1}, \qquad \sqrt{2}\,F_{r'} = P_{2r} - iP_{2r+1}, \tag{130}$$

we obtain

$$\left.\begin{aligned} \ddot{f}_r + 2\omega i\beta_{2r,\,2r+1}\dot{f}_r + \nu_r^2 f_r + 2\omega(\beta_{2r,\,-2r+2}\dot{f}_{-r+1} + \beta_{2r,\,-2r-2}\dot{f}_{-r-1}) &= F_{r'}, \\ \ddot{f}_{-r} + 2\omega i\beta_{-2r,\,-2r-1}\dot{f}_{-r} + 2\omega(\beta_{-2r,\,2r-2}\dot{f}_{r-1} + \beta_{-2r,\,2r+2}\dot{f}_{r+1}) &= 0, \end{aligned}\right\} \tag{131}$$

for all positive integral values of r, with a similar set of equations involving only the coordinates $f_{r'}$ and $f_{-r'}$. We therefore see that the set of coordinates f_r and f_{-r}, for all positive integral values of r, forms of itself a fundamental class, while the set of coordinates $f_{r'}$ and $f_{-r'}$ forms another fundamental class. In the corresponding modes of oscillation ξ will contain χ only in the factors $e^{in\chi}$ and $e^{-in\chi}$ respectively. We see also that f_{-r} and $f_{-r'}$ are coordinates of the *second class.*

On eliminating the coordinates of the second class from (131), we obtain

$$\begin{aligned} &\frac{4\omega\sigma^2\beta_{2r,\,-2r+2}\,\beta_{-2r+2,\,2r-4}}{\sigma + 2\omega\beta_{-2r+2,\,-2r+1}} f_{r-2} \\ &- \left\{\nu_r^2 - \sigma^2 - 2\omega\sigma\beta_{2r,\,2r+1} + \frac{4\omega^2\sigma\beta_{2r,\,-2r+2}^2}{\sigma + 2\omega\beta_{-2r+2,\,-2r+1}} + \frac{4\omega^2\sigma\beta_{2r,\,-2r-2}^2}{\sigma + 2\omega\beta_{-2r-2,\,-2r-3}}\right\} f_r \\ &+ \frac{4\omega^2\sigma\beta_{2r,\,-2r-2}\,\beta_{-2r-2,\,2r+4}}{\sigma + 2\omega\beta_{-2r-2,\,-2r-3}} f_{r+2} = F_{r'}, \end{aligned} \tag{134}$$

which is equivalent to the equation from which Hough* deduces his results, when the depth is uniform and the terms arising from the mutual gravitation of the elevated water are neglected.

On eliminating the coordinates of the first class from (131) in free motion, we obtain

$$\frac{4\omega^2\sigma\beta_{-2r,\,2r-2}\,\beta_{2r-2,\,-2r+4}}{\nu_{r-1}^2-\sigma^2-2\omega\sigma\beta_{2r-2,\,2r-1}}f_{-r+2}$$

$$-\left\{\sigma+2\omega\beta_{-2r,\,-2r-1}+\frac{4\omega^2\sigma\beta_{-2r,\,2r-2}^2}{\nu_{r-1}^2-\sigma^2-2\omega\sigma\beta_{2r-2,\,2r-1}}+\frac{4\omega^2\sigma\beta_{-2r,\,2r+2}^2}{\nu_{r+1}^2-\sigma^2-2\omega\sigma\beta_{2r+2,\,2r+3}}\right\}f_{-r}$$

$$+\frac{4\omega^2\sigma\beta_{-2r,\,2r+2}\,\beta_{2r+2,\,-2r-4}}{\nu_{r+1}^2-\sigma^2-2\omega\sigma\beta_{2r+2,\,2r+3}}f_{-r-2}=0. \qquad (134)'$$

When the rotation is slow, we see that to the third power of ω,

$$\sigma=-2\omega\beta_{-2r,\,-2r-1}+8\omega^3\beta_{-2r,\,-2r-1}\left\{\frac{\beta_{-2r,\,2r-2}^2}{\nu_{r-1}^2}+\frac{\beta_{-2r,\,2r+2}^2}{\nu_{r+1}^2}\right\}, \qquad (144)$$

for the oscillation of the second class in which f_{-r} is large compared with all other coordinates. The equation (144) is an instance of (69) of Part I.

---

* "On the Application of Harmonic Analysis . . .", Part II, *Phil. Trans. Roy. Soc.* (A), Vol. 191, p. 148 (1897); see also Love, "Notes on the Dynamical Theory of the Tides", *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 312 (1913).

THE SIMULTANEOUS SYSTEM OF TWO QUADRATIC QUATERNARY FORMS

By H. W. TURNBULL.

[Received September 20th, 1917.—Read November 1st, 1917.]

TABLE A.

The 125 Irreducible Forms of the System of Two Quaternary Quadratics $f = a_x^2$ and $f' = b_x^2$.

5 Invariants.	a_α^2, b_α^2, $(AB)^2$, a_β^2, b_β^2, *i.e.* Δ, Θ, Φ, Θ', Δ'.	§ 28 (ii)
5 Covariants.	f, $(A\beta x)^2$, $(Bax)^2$, f' and the quartic $a_x b_x a_\beta b_\alpha (AB)(A\beta x)(Bax)$.	§ 28 (ii) § 34
5 Contravariants.	u_α^2, $(Abu)^2$, $(Bau)^2$, u_β^2 and the quartic $u_\alpha u_\beta a_\beta b_\alpha (AB)(Abu)(Bau)$.	§ 34
16 Complexes.	$(Ap)^2$, $(Bp)^2$, $(abp)^2$, $(\alpha\beta p)^2$, $(AB)(Ap)(Bp)$,	§ 28 (ii)
	$(abp)(\alpha\beta p)\, a_\beta b_\alpha$,	§ 29
	$(abp)(a'bp)\, a_\beta a'_\beta$ $(\equiv F_1^2)$, $(abp)(\alpha b'p)\, b_\alpha b'_\alpha$ $(\equiv F_2^2)$	§ 14
	$(\alpha bp)(aa'p)(a''bp)\, a'_\beta a''_\beta$, $(aa'B)(Bp)(a''bp)(abp)\, a'_\beta a''_\beta$,	§§ 37
	$(aa'p)(\alpha\beta p)(abp)\, a'_\beta b_\alpha$, $(\alpha\beta p)(aa'B)(abp)\, a'_\beta b_\alpha (Bp)$,	to 39
	and 4 similar forms.*	
20 Mixed Forms containing x and u.	$a_x a_\beta u_\beta$, $b_x b_\alpha u_\alpha$, $(AB)(aBu)\, a'_x$, (§ 35),	§ 28 (ii)
	$(aBu)(a'Bu)\, a_x a'_x$,	§ 35
	$(Abu)(A\beta x)\, b_x u_\beta$, $(Bau)(Bax)\, a_x u_\alpha$,	§ 28 (iii)
	$(AB)(Abu)(Bau)\, a_x b_x$, $(AB)(A\beta x)(Bax)\, u_\alpha u_\beta$.	§ 29
	$(Abu)(A\beta x)\, b_\alpha a_\beta u_\alpha a_x$, $(Bau)(Bax)\, a_\beta b_\alpha u_\beta b_x$,	§ 31
	$(Abu)(A\beta x)\, b_\alpha u_\alpha u_\beta$ and a similar form	§ 29
	$(Abu)(A\beta x)\, a_\beta a_x b_x$,, ,,	,,
	$(AB)(Abu)(Bau)\, b_\alpha u_\alpha a_x$,, ,,	§ 31
	$(AB)(Abu)(Bau)\, a_\beta u_\beta b_x$,, ,,	,,
	$(a'a''bu)(Bau)(Ba'u)\, a_x b_x a''_x$,	§ 35
	$(a'a''\beta x)(Bax)(Ba'u)\, u_\alpha u_\beta a''_x$.	,,

* By "similar form" in the above is meant a form in which all the symbols a, A, α are interchanged respectively with b, B, β.

In this, $\alpha = aa'a''$, $A = aa'$, $(\alpha\beta p) = (\alpha\beta uv) = u_\alpha v_\beta - v_\alpha u_\beta$.

$\beta = bb'b''$, $B = bb'$, $(A\beta x) = a_\beta a'_x - a'_\beta a_x$.

TABLE A.—*Continued.*

Mixed forms containing x and p: 9 of orders 2 in x and 1 in p.	$(abp)\,a_a b_x$, $(a\beta p)\,a_\beta b_a a_x b_x$, $(AB)(Ep)(A\beta x)\,a_\beta a_x$,	§ 28 (ii), § 29β, § 29
	$(AB)(Ap)B\alpha x)\,b_a b_x$, $(AB)(A\beta x)(B\alpha x)(\alpha\beta p)$,	§ 29
	$(Ap)(A\beta x)\,a_\beta a_x$, $(Bp)(B\alpha x)\,b_a b_x$,	§ 28 (iii)
	$(abp)\,a'_\beta\,(A\beta x)\,b_x$, $(abp)\,b'_a\,(B\alpha x)\,a_x$.	§ 39
4 of orders 2 in x, 2 in p.	$(\alpha\beta p)(A\beta x)(Ap)\,b_a b_x$, and a similar form,	§ 29β
	$(abp)(AB)(B\alpha x)(Ap)\,b_a a_x$,,	§ 31
1 of orders 2 in x, 3 in p.	$(\alpha\beta p)(A\beta x)(B\alpha x)(Ap)(Bp)$.	§ 29
Mixed forms containing u and p. Correlative of the above $9+4+1=14$ forms.	Re-write above set with u for x, a ,, a, a ,, α, β ,, b, b ,, β, and A, B, p unchanged.	§ 29
Mixed forms containing x, p, u: 16 linear in all variables.	$(Abu)(Ap)\,b$, $(A\beta x)(Ap)\,u_\beta$, and two similar forms,	§ 28(ii)
	$(abp)\,a_x b_a u_a$, $(a\beta p)\,a_\beta a_x u_\alpha$, ,, ,,	§ 28 (iii)
	$(AB)(Abu)(Bp)\,b_x$, $(AB)(A\beta x)(Bp)\,u_\beta$, ,, ,,	
	$(abp)(Abu)(A\beta x)\,a_\beta$, and one similar form,	§ 29
	$(\alpha\beta p)(Abu)(A\beta x)\,b_\alpha$, ,, ,,	§ 29
26 of order 2 in p, linear in u and x.	$(Ap)(Bp)(Abu)\,b'_x$, $(AB)(\alpha\beta p)(Ap)(Bau)\,a_\beta b_a b_x$,	§ 35, § 34
	$(abp)(\alpha\beta p)\,a_\beta u_a b_x$, $(abp)(\alpha\beta p)\,b_\alpha u_\beta a_x$,	§ 29β
	$(abp)(Abu)(Ap)\,a_x$, $(\alpha\beta p)(A\beta x)(Ap)\,u_\alpha$,	§ 28 (iii)
	$(AB)(abp)(Abu)(Bp)\,a_x$, $(AB)(\alpha\beta p)(A\beta x)(Bp)\,u_\alpha$,	§ 29
	$(\alpha\beta p)(Abu)(Ap)\,a_\beta b_\alpha a_x$,	§ 31
	and five similar forms.	
	$(abp)\,a'_\beta b_x u_\beta\,(Ap)$, $(abp)\,a'_\beta\,(Abu)(A\beta x)(A'p)$,	§ 37, § 38
	$(AB)(abp)\,a'_\beta b_x u_\beta\,(Bp)$, $(abp)\,a'_\beta(Abu)\,(A\beta x)(A'B)(Bp)$,	,,
	$(abp)\,a'_\beta\,(a''bp)\,a''_\beta\,(Ab'u)\,b'_x$, $(abp)\,a'_\beta b_\alpha\,(\alpha\beta p)(A\beta' x)\,u_{\beta'}$,	§ 38
	and six similar forms.	
4 forms of orders (2, 1, 2).	$(Abu)(Bau)(Ap)\,b'_x a_x$, $(Abu)(B\alpha x)(Ap)\,b'_x u_\alpha$,	§ 35
	and two similar forms.	

I.

Preliminary Reductions.

1. In the *Mathematische Annalen*, Bd. 56, Gordan has worked out a system of invariants for two quaternary quadratics. In the following pages it is proposed to shew how Gordan's system can be very much

simplified. In fact, we shall reduce the system to 125 forms instead of Gordan's 580 forms.

The theorems established in Sections I and II of this paper are substantially the same as in Chapters 1 and 2 of Gordan's work. They are only reproduced here to make the investigation complete.

2. Let
$$f = a_x^2 = a_x'^2 = \dots \Big\}$$
and
$$f' = b_x^2 = b_x'^2 = \dots \Big\}, \tag{1}$$
be the two quadratics.

Then any invariant is expressible as a product of factors d_x, (dd_1p), (dd_1d_2u), $(dd_1d_2d_3)$, where d stands for a or b and each different d occurs exactly twice in the form. The symbols x, p, u are the variables,

u denoting plane coordinates,
$p = (uv)$ „ line „ ,
$x = (uvw)$ „ point „ ,

so that $u_x = v_x = w_x = 0$ always holds good.

To be more precise, we define v by the relations
$$u_iv_k - v_iu_k = p_{ik} \quad (i, k = 1, 2, 3, 4).$$

Any invariant J is then a function of the variables x, p, u: but for purposes of reduction it is sometimes well to break up p into its elements u, v; in which case we may have, in addition to the four types of factors d_x, (dd_1p), ... above, the new factor
$$[(dd_1d_2u)(d_3d_4d_5v) - (dd_1d_2v)(d_3d_4d_5u)].$$
This factor is equal to
$$(dd_1d_2d_3)(d_4d_5uv) + (dd_1d_2d_4)(d_5d_3uv) + (dd_1d_2d_5)(d_3d_4uv),$$
which contains p explicitly.

The Fundamental Identities of Quaternary Forms.

3. Neglecting reducible terms containing u_x or v_x, we have three identities
$$(abcd)\,e_x = (abce)\,d_x + (abed)\,c_x + (aecd)\,b_x + (ebcd)\,a_x, \tag{I}$$
$$(abcu)\,d_x = (abdu)\,c_x + (adbu)\,b_x + (dbcu)\,a_x, \tag{II}$$
$$(abp)\,c_x = (acp)\,b_x + (cbp)\,a_x. \tag{III}$$

In these, let g_i denote the bracket in the left-hand member. Then a useful reference symbol is

$(\dot{g}_i,\ \dot{e}_x)$ for identity (I),

$(\dot{g}_i,\ \dot{d}_x)$ „ (II),

$(\dot{g}_i,\ \dot{c}_x)$ „ (III).

These identities follow from the identical vanishing of a determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_x \\ b_1 & \dots & \dots & & \\ c_1 & \dots & \dots & & \\ d_1 & \dots & \dots & & \\ e_1 & \dots & \dots & & \end{vmatrix}.$$

Determinantal Permutations.

4. Let a_δ denote a_x or else a bracket containing a. Then, if a_δ, b_ϵ is a pair of such symbolic factors, let

$$\dot{a}_\delta \dot{b}_\epsilon \text{ denote } a_\delta b_\epsilon - b_\delta a_\epsilon ;$$

but let $(\dot{a}\dot{b}kl)$ leave $(abkl)$ unchanged.

This is a determinantal permutation of a, b. We extend the operation to any number μ of such symbols, $a, b, c, \dots$, the dotted letters undergoing all interchanges, as in the development of a determinant. Thus

$$\dot{a}_\delta \dot{b}_\epsilon \dot{c}_\zeta \text{ denotes the 6 terms } a_\delta b_\epsilon c_\zeta - a_\delta c_\epsilon b_\zeta + \dots,$$

while $(\dot{a}\dot{b}kl)\dot{c}_\delta$ „ 3 „ $(abkl)c_\delta + (bckl)a_\delta + (cakl)b_\delta$.

This allows a very concise notation for the symbolic identities. In fact,

if $\mu = 2$, we have the identity $(\dot{a}kmn)\dot{b}_\delta = (ab\dot{m}\dot{n})\dot{k}_\delta$; (IV)

„ $\mu = 3$, „ , besides this, $(\dot{a}kmn)\dot{b}_\delta\dot{c}_\epsilon = (abc\dot{n})\dot{k}_\delta\dot{m}_\epsilon$, (V)

with its 6 terms on each side ;

if $\mu = 4$, we have $\dot{a}_\delta\dot{b}_\epsilon\dot{c}_\zeta\dot{d}_\eta = (abcd)(\delta\epsilon\zeta\eta)$

$= (abcd)\dot{d}_\epsilon\dot{d}'_\zeta\dot{d}''_\eta$, where $\delta = (dd'd'')$, (VI)

$(\dot{a}\dot{b}mn)\dot{c}_\delta\dot{d}_\epsilon = (abcd)\dot{m}_\delta\dot{n}_\epsilon$, (VII)

$(\dot{a}\dot{b}kl)(\dot{c}\dot{d}mn) = (abcd)(klmn)$; (VIII)

if $\mu > 4$, the result is zero, *e.g.*

$$(\dot{a}\dot{b}\dot{c}\dot{d})\,\dot{e}_x = 0.$$

Each identity can easily be verified,* being deduced from Identity (I).

II.

Application of these Identities to Two Quadrics.

5. Suppose that in the above identities, $a, b, c, \ldots$ refer to one quadric f. Complementary to these there will be another set of symbols $a, b, c, \ldots$ in each symbolic product P representing an invariant of the system. Now let these μ complementary symbols lie in one bracket, g, where $\mu \geqslant 4$. We may then typify the product P as

$$(abc \ldots kd_1d_2 \ldots d_{4-\mu})[a, b, c, \ldots, k],$$

Now permute $[a, b, c, \ldots, k]$ determinantally. Then from one or other of the identities, we have

$$(abc \ldots kd_1d_2 \ldots d_{4-\mu})[\dot{a}, \dot{b}, \dot{c}, \ldots, \dot{k}] = \Sigma g g_1 g_2 \ldots,$$

where g_1 is a bracket containing $(abc \ldots k \ldots)$, and where g denotes the initial bracket on the left. But each term of the series on the left may immediately be reduced back to $+P$, since all the symbols $a, b, \ldots, k$ are equivalent. We may therefore say that

$$P \equiv 0 \mod gg_1,$$

where both g and g_1 contain $(abc \ldots k)$.

Moreover we may select for g_1, any bracket whatever, except g, of the original form P. What happens is that the contents of g_1 are diverted to the positions in P which were originally held by the μ symbols $a, b, \ldots, k$.

Suppose now $\mu = 4$. Then gg_1 is equal to the invariant $(abcd)^2$. Hence if P contain

$$(aa_1a_2a_3) \quad \text{or} \quad (bb_1b_2b_3),$$

it is reducible. (2)

---

* For a formal proof in the general case, see the author's paper, "Quadratics in n Variables", pp. 201–4, *Camb. Phil. Trans.*, Vol. 21, No. 8.

The Characters $c_1, c_2, c_{12}, \ldots$.

6. Let $c_1 =$ the degree of P in the coefficients of f,

$c_2 =$,, ,, f',

$c_{1\nu} =$ the total of factor pairs gg_1 containing $a_1 a_2 \ldots a_\nu$,

$c_{2\nu} =$,, ,, $b_1 b_2 \ldots b_\nu$.

It follows that, if $c_{14} > 0$ or $c_{24} > 0$, P is reducible.

Then we consider the forms P in the following order :—

1. P_1 comes before P_2 if c_1 or c_2 in P_1 is less than in P_2, while the other c_2 or c_1 is not greater.

2. If P_1 and P_2 have the same c_1 and c_2, take P_1 before P_2 if c_{13} is greater in P_1, while c_{23} is not less ; or *vice versa* for c_{13}, c_{23}.

3. If c_1, c_2, c_{13}, c_{23} are the same, take P_1 before P_2, if c_{12} is greater in P_1, while c_{22} is not less ; or *vice versa* for c_{12}, c_{22}.

The Modular Notation.

7. If P contains a factor g which implies reducibility, we write either $P \equiv 0$, or $g \equiv 0$, indifferently.

There may be groups of equivalent forms $P_1, P_2, P_3, \ldots$, which are such that each form of the group differs from another by reducible terms. Then we write

$$P_1 \equiv P_2 \equiv P_3 \ldots .$$

The Contracted Symbolic Notation.

8. Let a denote three equivalent symbols aa_1a_2 bracketed twice.

β ,, ,, bb_1b_2 ,,

A two ,, aa_1 ,,

B ,, ,, bb_1 ,,

Then c_{13} denotes the number of different symbols α in the form. And it is clear from § 5 that

$$a_\alpha \equiv 0, \quad b_\beta \equiv 0. \tag{3}$$

The Symbols α, β.

9. If P contains $(a_1a_2a_3u)(a_ia_ja_kv)$, where $i, j, k = 1, 2, 3$ or any other suffix, P is reducible.

For, if the complementary a_1 is absent from the second bracket g_2, by using the process $(\dot{g}_2, \dot{a}_1)$ of § 3, we express this as 3 terms

$$(a_1a_2a_3u)(a_1a_ja_kv) \bmod (a_ia_ja_ka_1), \text{ i.e. } c_{14}.$$

Repeating the process, twice if necessary, we obtain

$$(a_1a_2a_3u)(a_1a_2a_3v) \bmod c_{14},$$

i.e.
$$u_\alpha v_\alpha.$$

As this increases c_{13}, with c_1c_2 unaltered, P is therefore reducible. Further, it implies that all the symbols a, a_1, a_2 of P are interchangeable in every way whatever. We may therefore drop the suffix 1, 2, ..., and call them all a. Similarly for β.

In the above, u and v denote any symbol.

The Factor $(a_1a_2a_3d) \equiv 0$.

10. If $a_1a_2a_3$ are not twice bracketed, their complements are found either in brackets or in factors a_{1_β}, a_{1_x}. If the tag suffix is the same in two cases, P is zero.

Hence there is at least one bracket g_1. By a determinantal permutation we may bracket the three complements $a_1a_2a_3$ in g_1. This expresses P in terms of products where a can be written twice for $(a_1a_2a_3)$. So P is reduced. But we must include a new type of bracket defined as

$$(\alpha\beta p) = (u_\alpha v_\beta - v_\alpha u_\beta) = \dot{u}_\alpha \dot{v}_\beta. \tag{4}$$

For if in the above reduction g_1 contained p or (uv), then necessarily uv are separated when $a_1a_2a_3$ all come into g_1. Let them stand in $u_\delta v_{\delta'}$. Since uv started from one bracket g_1, they will now be found in the combination

$$\dot{u}_\delta \dot{v}_{\delta'},$$

where δ, δ' stand for α, β, x or (dp), dd_1u, dd_1d_2. In these cases uv may be rebracketed in one or other factor u_δ or $v_{\delta'}$, thereby regaining the lost symbol p, unless both δ, δ' are α, β or x. But u_x, v_x are both 0, which only leaves the new type $(\alpha\beta p)$.

To sum up the present results, we must consider forms made of factors

$$u_\alpha,\ u_\beta,\ a_x,\ b_x,\ a_\beta,\ b_\alpha,\ (\alpha\beta p),\ (abp),\ (aa_1p),\ (aa_1bu),\ (bb_1p),\ (abb_1u),\ (aa_1bb_1). \tag{5}$$

Factors g with two Symbols a or b.

11. The product $(aa_1kl)(a_ia_jmn)$, where $k, l, m, n = u, b,$ or $p,$ is reducible. As before, by permuting the complements of $a, a_1,$ we express this in terms of

$$(aa_1kl)(aa_1mn),$$

and terms with more than two symbols a in the second bracket. We write this as

$$(Akl)(Amn).$$

Thus if there are an even number of brackets each with two a's, we can pair off symbols $AA, A_1A_1, \ldots$ to fit them. Clearly the permutation A, A_1 would now shew that any pair of A's can be interchanged. The same can be done for B. We have then possibly one pair (aa_1) left over, which we do not call A until its complements are explicitly bracketed. As for the other A's, we may drop their suffixes. We deal with (bb_1) in like manner.

We have now to consider these factors

$$(aa_1bb_1), (abb_1u), (baa_1u), (abp), (AB), (aBu), (bAu), (Ap), (Bp), a_\beta, \text{ etc.} \quad (6)$$

The Factor (aa_1ij).

12. If P contain this odd factor (aa_1ij), the complements a, a_1 must occur in factors of type

$$(abb_1u), (abp), (aBu), a_\beta, a_x, \text{ or } g_1, g_2, g_3, g_4, g_5 \text{ say.}$$

If g_1 occurs, we may bracket the complements a, a_1 in g_1 and reduce P. The same process avails if a, a_1 occur in two factors like g_2, or g_2g_5 or g_3g_4, or g_3g_3. Two factors g_4g_4 or g_5g_5 are clearly reducible. We have then left over

$$g_2g_3,\ g_2g_4,\ g_3g_5,\ g_4g_5.$$

Now
$$\begin{aligned} g_2g_3 &= (abp)(a_1Bu) \\ &= \tfrac{1}{2}(\dot{a}bp)(\dot{a}_1Bu) \quad (\text{since } \overline{aa_1} \text{ is in another bracket}) \\ &= \tfrac{1}{2}\Sigma(b''bp)(Ab'u) \quad (\text{if } B = b'b'') \\ &= \tfrac{1}{2}\Sigma(Bp)(Abu) \quad (\S\ 11) \\ &= 0 \quad (\text{for } c_{12} \text{ is increased}). \end{aligned}$$

We have left over g_2g_4, g_3g_5, g_4g_5, *i.e.* forms containing

$$(aa_1ij)(\dot{a}bp)\,\dot{a}_{1_\beta}, \quad (aa_1ij)(\dot{a}Bu)\,\dot{a}_{1_x}, \quad (aa_1ij)\,\dot{a}_\beta\dot{a}_{1_x}.$$

Let us denote these by

$$\left.\begin{aligned} F_1 &= -(\dot{a}bp)\,a_{1_\beta}, \\ (AB)' &= (\dot{a}Bu)\,\dot{a}_{1_x} = -(\dot{b}Au)\dot{b}_{1_x} \text{ [by (I) of § 3]}, \\ \text{and} \qquad (A\beta x) &= \dot{a}_\beta \dot{a}_{1_x}. \end{aligned}\right\} \tag{7}$$

13. Similarly for an odd factor (bb_1ij), only now we have to add to the five possibilities corresponding to $g_1, g_2, \ldots, g_5$, a sixth

$$g_6 = F_1 = -(\dot{a}bp)\,\dot{a}_{1_\beta}.$$

If b, b_1 stand in F_1 and another p-bracket, or F_1 and b_{1_x}, we may at once bracket bb_1 in F_1 and reduce it. The other two cases are

$$F_1(b_1Au) \quad \text{and} \quad F_1 b_{1_\alpha}.$$

Of these, the first is reduced as in g_2g_3 just above. The second is substantially

$$\begin{aligned} P &= (aa_1ij)(bb_1kl)(abp)\,a_{1_\beta}b_{1_\alpha} \quad (\text{where } p = uv) \\ &= \tfrac{1}{2} \;\;,, \quad ,, \quad (a\dot{b}p)\,a_{1_\beta}\dot{b}_{1_\alpha}, \\ &\equiv \tfrac{1}{2} \;\;,, \quad ,, \quad (abb_1\dot{v})\,a_{1_\beta}\dot{u}_\alpha \quad \text{[by (I) of § 3]} \\ &\equiv \tfrac{1}{4} \;\;,, \quad (Bkl)(\dot{a}B\dot{v})\,\dot{a}_{1_\beta}\dot{u}_\alpha \end{aligned}$$

(where the dots go in two pairs, aa_1 being permuted separately from uv)

$$\begin{aligned} &\equiv \tfrac{1}{4} \;\;,, \quad ,, \quad (AB)\,\dot{v}_\beta\dot{u}_\alpha \\ &\equiv \tfrac{1}{4}(Aij)(Bkl)(AB)(\alpha\beta p) \\ &\equiv 0. \end{aligned} \tag{8}$$

14. This completes the preliminary investigation. All the pairs aa_1, bb_1 are replaced by A, B.

Any form P of the system is composed of factors of types (ϕ), $(\phi\psi)$, and F; where

$$(\phi) = a_x,\ b_x,\ u_\alpha,\ u_\beta,\ (Ap),\ (Bp),$$

$$(\phi\psi) = (abp),\ a_\alpha,\ a_\beta,\ b_\alpha,\ b_\beta,\ (aBu),\ (bAu),\ (AB),$$

$$(AB)',\ (A\beta x),\ (B\alpha x),\ (\alpha\beta p).$$

$$F_1 = (bap)\,a'_\beta - (ba'p)\,a_\beta = (\dot{b}\dot{a}p)\,\dot{a}'_\beta = (A\beta\overline{\dot{b}\dot{p}}),$$

$$F_2 = (abp)\,b'_\alpha - (ab'p)\,b_\alpha = (a\dot{b}p)\,\dot{b}'_\alpha = (B\alpha\overline{\dot{a}\dot{p}}).$$

Moreover, each symbol can be interchanged with any equivalent symbol, a with a', A with A', and so on.

Gordan proceeds to arrange the system in 6 classes J.

$J^{(1)}$	contains the squares of these factors including $(AB)(AB)'$	21	forms
$J^{(2)}$	,, only factors $(\phi\psi)$	23	,,
$J^{(3)}$	,, ,, $(\phi\psi)$, (ϕ)	186	,,
$J^{(4)}$	,, F_1 but not F_2	134	,,
$J^{(5)}$	,, F_2 ,, F_1	134	,,
$J^{(6)}$	,, F_1F_2	82	,,
	Total.....	580	,,

We shall prove that these sets contain at most

$$21+7+73+12+12+0 = 125$$

irreducible forms.

III.

The 1, 2, 3 *Notation.*

15. To discriminate between forms which are not equivalent it is well to change our notation, using a device employed in dealing with systems of ternary quadratics.*

Let every form P be rewritten according to the following notation :—

for a_x write 1_ξ or ${}_\xi 1$, for b_x write ω_1 or ${}_1\omega$,

,, (Ap) ,, 2_ξ ,, ${}_\xi 2$, ,, (Bp) ,, ω_2 ,, ${}_2\omega$,

,, u_α ,, 3_ξ ,, ${}_\xi 3$, ,, u_β ,, ω_3 ,, ${}_3\omega$:

for (abp) write 1_1 or ${}_1 1$,

,, (aBu) ,, 1_2 ,, ${}_2 1$,

,, (AB) ,, 2_2 ,, ${}_2 2$,

,, $(AB)'$,, $2_2'$,, ${}_2 2'$,

,, $(a\beta p)$,, 3_3 ,, ${}_3 3$,

and so on.

Thus 1, 2, 3 denote a, A, α, while 1, 2, 3 used as suffices denote b, B, β, respectively.

* Cf. Turnbull, "Ternary Quadratic Types", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 83.

The factors F_1 and F_2 are however retained.

Let the symbols $i, j, k, \ldots$ denote 1, 2, 3 indiscriminately.

*Chains and Tags.**

Further, let such a product as

$$i_j \,.\, k_j \,.\, k_l \,.\, m_l \ldots$$

be abbreviated to $\begin{pmatrix} \ldots i\ k\ m \ldots \\ j\ \ l \end{pmatrix}$.

If this expression start with i and end with ξ or ω, it is called a *tag*. We denote it by (i). Otherwise it is a *chain*, when both the extreme symbols are of type ξ or ω, or else neither ξ, ω appear at all. We denote a chain by its extreme symbols

$$(i, j), \quad \begin{pmatrix} i, \\ j \end{pmatrix}, \quad \begin{pmatrix} \omega \\ \xi, \end{pmatrix}, \quad \text{etc.}$$

Properties of Chains and Tags.

16. (1) In one chain or tag all the upper symbols i, k, m must differ, except possibly the two extremes of a chain.

(2) So also must all the lower symbols j, l differ. Otherwise the form is reducible. Thus the chain

$$\begin{pmatrix} 1\ 3\ 1\ \omega \\ \xi\ 2\ 1\ 3 \end{pmatrix} \text{ contains the factor } \begin{pmatrix} 1\ 3\ 1 \\ 2\ 1 \end{pmatrix}.$$

(3) In a tag (i), each symbol j, k, ... is paired, but i only occurs once.

(4) In a chain (ξ, ξ) or (ω, ω) each symbol is paired.

(5) In a chain (ξ, ω), (i, j), or $\begin{pmatrix} i, \\ j \end{pmatrix}$, each symbol except the two extremes is paired.

(6) The symbols ξ, ω can only occur as extreme symbols.

* See note on previous page.

The Set of Forms $J^{(1)}$, $J^{(2)}$, $J^{(3)}$.

17. It is obvious that all forms P which are made up of symbols i_ξ, $_j\omega$, k_l, but which have no factor F_1 or F_2, are only of two kinds, viz. :—

(I) chains such as $\begin{pmatrix} i & \dots \\ \xi & & \xi \end{pmatrix}$, $\begin{pmatrix} i & \dots & \omega \\ \xi & & \end{pmatrix}$, $\begin{pmatrix} \omega & \dots & \omega \\ i & & \end{pmatrix}$.

(II) chains such as $\begin{pmatrix} i & k & \dots & i \\ j & & & \end{pmatrix}$, $\begin{pmatrix} k & \dots & j \\ j & & \end{pmatrix}$.

For P consists of chains and tags only, and each extreme symbol of a chain or tag must be paired with some other extreme symbol elsewhere. The resultant of such can only be one chain.

Reducibility.

18. If by any process we can transform P to terms involving (1) either less symbols 1, 2, 3, or (2) more symbols ξ, ω, then P is reducible.

Proof.—(1) The number of pairs of symbols 1, 2, or 3 in P is the grade of P. To diminish the number of these symbols is to lower the grade. All we have to do is to deal with the forms in ascending value of grade.

(2) If we consider the chains (I) before chains (II) of the same grade, then P is reduced if we increase the number of ξ, ω symbols in P.

IV.

Formulae of Reduction.

19. The above principles will now be illustrated by establishing the following formulæ :—

(A) $2'_2 \cdot 3_1 \equiv 2_1 \cdot 3_2 + 2_2 \cdot 3_\xi \cdot {}_1\omega$,

$2'_2 \cdot 1_3 \equiv -2_3 \cdot 1_2 - 2_2 \cdot 1_\xi \cdot {}_3\omega$,

(B) $2'_2 \cdot 1_1 \equiv -2_1 \cdot 1_\xi \cdot {}_2\omega - 1_2 \cdot {}_1\omega \cdot 2_\xi$,

$2'_2 \cdot 3_3 \equiv +2_3 \cdot 3_\xi \cdot {}_2\omega + 3_2 \cdot 2_\xi \cdot {}_3\omega$,

(C) $F_2 \cdot {}_3\omega \equiv 3_3 \cdot 1_2 + 1_3 \cdot 3_\xi \cdot {}_2\omega$,

$F_1 \cdot 3_\xi \equiv -3_3 \cdot 2_1 + 3_1 \cdot 2_\xi \cdot {}_3\omega$,

(D) $F_2 \cdot {}_1\omega \equiv -1_1 \cdot 3_2 - 3_1 \cdot 1_\xi \cdot {}_2\omega,$

$F_1 \cdot 1_\xi \equiv 1_1 \cdot 2_3 - 1_3 \cdot 2_\xi \cdot {}_1\omega,$

(E) $F_2 \cdot 2_1 \equiv 2_\xi \cdot 3_1 \cdot 1_2 + 1_1 \cdot 3_\xi \cdot 2_2,$

$F_1 \cdot 1_2 \equiv \omega_2 \cdot 1_3 \cdot 2_1 - 1_1 \cdot \omega_3 \cdot 2_2,$

(F) $F_2 \cdot 2_3 \equiv -2_2 \cdot 3_3 \cdot 1_\xi - 2_\xi \cdot 1_3 \cdot 3_2,$

$F_1 \cdot 3_2 \equiv 2_2 \cdot 3_3 \cdot \omega_1 - \omega_2 \cdot 3_1 \cdot 2_3,$

(G) $2_1 \cdot 1_2 \cdot 3_3 \equiv 0,$

(H) $2_3 \cdot 3_2 \cdot 1_1 \equiv 0,$

(J) $F_1 F_2 \equiv -2_2 \cdot 3_3 \cdot 1_1 + 2_\xi \cdot {}_2\omega \cdot 1_3 \cdot 3_1,$

(K) $F_2 \cdot 2'_2 \equiv +2_\xi \cdot 3_2 \cdot 1_2 - 2_2 \cdot 3_\xi \cdot {}_2\omega \cdot 1_\xi,$

$F_1 \cdot 2'_2 \equiv 0,$

(L) $2_1 \cdot 3_3 \cdot 1_\xi + 1_1 \cdot 3_\xi \cdot 2_3 \equiv 2_\xi \cdot 3_\xi \cdot 1_3 \cdot {}_1\omega + 1_\xi \cdot {}_3\omega \cdot 3_1 \cdot 2_\xi,$

(M) $1_2 \cdot 3_3 \cdot {}_1\omega + 1_1 \cdot {}_3\omega \cdot 3_2 \equiv 0.$

Proofs of the Formulae.

(A).

20. $2_1 \cdot 3_2 = (Abu)(Bax) = (Abu)\, \dot{b}'_a \dot{b}''_x$ (say)

$\equiv (Ab\dot{b}')\, u_a \dot{b}''_x + (A\dot{b}'u)\, b_a \dot{b}''_x \pmod{a_a}$ (by 1d. 1)

$\equiv (Ab''b')\, u_a b_x + (AB)'\, b_a$

(and terms with $bb'b''$ bracketed, which are reducible by § 10)

$\equiv -(AB)\, u_a \cdot b_x + (AB)'\, b_a$

$\equiv -2_2 \cdot 3_\xi \cdot {}_1\omega + 2'_2 \cdot 3_1.$

Interchange the symbols of the two quadratics f and f' throughout, and the second formula is established. The signs are changed because

$$(BA)' = -(AB)'.$$

(B).

21. $2'_2 \,.\, 1_1 = (aa'\dot{b}u)\,\dot{b}'_x\,(a''b''p)$;

and using 1d. II, remembering $(up) = 0$, this

$$= (aa'b''u)\,\dot{b}'_x\,(a''\dot{b}p) + (\dot{a}b''\dot{b}u)\,\dot{b}'_x\,(a''\dot{a}'p)$$

(where a is only permuted with a' and b with b')

$$\equiv (Ab''u)\,a''_x\,(b'bp) + (a''b''\dot{b}u)\,\dot{b}'_x\,(aa'p)$$

$$\equiv \quad ,, \quad + (a''b'bu)\,b''_x\,(aa'p)$$

$$\equiv -2_1 \,.\, 1_\xi \,.\, {}_2\omega - 1_2 \,.\, {}_1\omega \,.\, 2_\xi .$$

The second formula (B) follows from the first by the principle of duality, which is effected by interchanging u with x, a with α, and b with β: or simply 1 with 3. But we give a direct proof: we have

$$2'_2 \,.\, 3_3 = -(\dot{a}Bu)\,\dot{a}'_x\,(u_\alpha v_\beta - u_\beta v_\alpha).$$

Using 1d. I upon (aBu), v_β, this product

$$\equiv -(vBu)\,\dot{a}'_x\dot{u}_\alpha\dot{a}_\beta - (\dot{a}Bv)\,\dot{a}'_x u_\alpha u_\beta + (\dot{a}Bu)\,\dot{a}'_x u_\beta v_\alpha .$$

The first term $\equiv (Bp)\,\dot{a}'_x\dot{a}_\beta u_\alpha$ [since $p = (uv)$],

$$\equiv (Bp)(A\beta x)\,u_\alpha \quad (i.e.\ {}_2\omega \,.\, 3_\xi \,.\, 2_3).$$

The other two terms $= -(\dot{a}B\dot{v})\,\dot{a}'_x\dot{u}_\alpha u_\beta$ (as they stand)

$$\equiv -(aa'\dot{b}'\dot{v})\,\dot{b}_x\dot{u}_\alpha u_\beta \ \text{(since } v_x = 0);$$

and now using 1d. I on u_α and this bracket, this

$$\equiv -(aa'uv)\,\dot{b}_x\dot{b}'_\alpha u_\beta$$

$$\equiv \quad (Ap)(Bax)\,u_\beta \quad (i.e.\ 2_\xi \,.\, 3_2 \,.\, {}_3\omega).$$

Hence $2'_2 \,.\, 3_3 \equiv {}_2\omega \,.\, 3_\xi \,.\, 2_3 + 2_\xi \,.\, 3_2 \,.\, {}_3\omega.$

(C).

22. $F_2 \,.\, {}_3\omega = (a\dot{b}p)\,\dot{b}'_\alpha \,.\, u_\beta = (a\dot{b}uv)\,\dot{b}'_\alpha \,.\, u_\beta$

$$\equiv (abb'v)\,u_\alpha u_\beta - (abb'u)\,v_\alpha u_\beta .$$

Now using 1d. I upon the first term, this

$$\equiv (ubb'v)\, u_\alpha a_\beta + (abb'u)\, u_\alpha v_\beta - (abb'u)\, v_\alpha u_\beta$$

$$\equiv (Bp)\, u_\alpha a_\beta + (aBu)(a\beta p)$$

$$\equiv {}_2\omega \,.\, 3_\xi \,.\, 1_3 + 1_2 \,.\, 3_3.$$

Interchange f and f', and the second formula follows.

(D).

These are the reciprocals of (C) :

$$F_2 \,.\, {}_1\omega = (a\dot{b}p)\, \dot{b}'_a b''_x$$

$$\equiv (b''\dot{b}p)\, \dot{b}'_a a_x + (ab''p)\, \dot{b}'_a \dot{b}_x \quad \text{(by 1d. III)}$$

$$\equiv (b'bp)\, b''_a a_x + (ab''p)(Bxa) \ [\text{mod } (bb'b'')]$$

$$\equiv -(Bp)\, b''_a . a_x - (ab''p)(Bax)$$

$$\equiv -\omega_2 \,.\, 3_1 \,.\, 1_\xi - 1_1 \,.\, 3_2.$$

The second formula is analogous to this.

(E).

23. $$F_2 \,.\, 2_1 = (a\dot{b}p)\, \dot{b}'_a (b''Au)$$

$$\equiv (a\dot{b}p)\, b''_a (\dot{b}'Au) + (a\dot{b}p)\, u_a (b''A\dot{b}')$$

$$\equiv (a\dot{a}'p)\, b''_a (b'b\dot{a}''u) + (ab''p)\, u_a (bAb')$$

[on collecting bb' into the second bracket and omitting reducing factors $(b''B)$ or (aA), A being $(a'a'')$]

$$\equiv (a''a'p)\, b''_a (b'bau) + (ab''p)\, u_a (Abb')$$

$$\equiv (Ap)\, b''_a (aBu) + (ab''p)\, u_a (AB)$$

$$\equiv 2_\xi \,.\, 3_1 \,.\, 1_2 + 1_1 \,.\, 3_\xi \,.\, 2_2.$$

Similarly for $F_1 \,.\, 1_2$.

(F).

These are the reciprocals of (E):

$$\begin{aligned} F_2 \,.\, 2_3 &= (a\dot{b}p)\,\dot{b}'_\alpha \,.\, \dot{a}'_\beta \dot{a}''_x \\ &\equiv (\dot{a}''\dot{b}p)\,\dot{b}'_\alpha \dot{a}'_\beta a_x + (a\dot{a}''p)\,\dot{b}'_\alpha \dot{a}'_\beta \dot{b}_x \quad \text{(by 1d. III)} \\ &\equiv -(aa'bb')(\alpha\beta p)\,a_x + (a'a''p)\,\dot{b}'_\alpha a_\beta \dot{b}_x \quad [\text{cf. § 13, (8)}] \\ &\equiv -2_2 \,.\, 3_3 \,.\, 1_\xi - 2_\xi \,.\, 3_2 \,.\, 1_3. \end{aligned}$$

Similarly for $F_1 \,.\, 3_2$.

(G).

24. Multiply the first formulæ (C) and (E) by 2_1 and ${}_3\omega$ respectively and equate results. This expresses $2_1 \,.\, 1_2 \,.\, 3_3$ in terms each involving ξ, ω symbols, and so it is reducible (§ 18).

Likewise (H) follows from formulæ (D) and (F).

(J).

25. Here

$$\begin{aligned} F_1 F_2 &= (a\dot{b}p)\,\dot{b}'_\alpha\,(b''\dot{a}'p)\,\dot{a}''_\beta \\ &\equiv (\dot{a}'\dot{b}p)\,\dot{b}'_\alpha\,(b''ap)\,\dot{a}''_\beta + (a\dot{a}'p)\,\dot{b}'_\alpha\,(b''\dot{b}p)\,\dot{a}''_\beta \\ &\equiv -(AB)(\alpha\beta p)(abp) + (Ap)(Bp)\,b_\alpha a_\beta \quad [\text{as in formula (F)}] \\ &\equiv -2_2 \,.\, 3_3 \,.\, 1_1 + 2_\xi \,.\, 2\omega \,.\, 1_3 \,.\, 3_1. \end{aligned}$$

(K).

$$\begin{aligned} 26. \quad F_2 \,.\, 2'_2 &\equiv (a\dot{b}p)\,\dot{b}'_\alpha\,(\dot{a}''Bu)\,\dot{a}'_x \\ &\equiv (\dot{a}''\dot{b}p)\,\dot{b}'_\alpha\,(aBu)\,\dot{a}'_x + (a\dot{b}u\dot{a}'')\,\dot{b}'_\alpha\,(vBu)\,\dot{a}'_x \quad \text{(by 1d. I)} \\ &\equiv (a''a'p)\,\dot{b}'_\alpha\,(aBu)\,\dot{b}_x + (ab\dot{b}'\dot{a}'')\,u_\alpha\,(Bvu)\,\dot{a}' \\ &\equiv 2_\xi \,.\, 1_2 \,.\, 3_2 + (a'bb'a'')\,u_\alpha\,(Bvu)\,a_x \\ &\equiv 2_\xi \,.\, 1_2 \,.\, 3_2 - 2_2 \,.\, 3_\xi \,.\, {}_2\omega \,.\, 1_x. \end{aligned}$$

Similarly for $F_1 \,.\, 2'_2$.

(L).

27. This follows by combining formulæ (C) and (D), eliminating F_1 as in the case of (H). In formula (L) the two right members contain more ξ, ω factors than the left. They are therefore reducible. Lastly (M) follows by eliminating F_2 from (C) and (D).

V.

The Complete System.

28. We take the forms in the following order :—

(1) forms with no factor $2_2'$, F_1, F_2,

(2) ,, $2_2'$ but no F_1, F_2,

(3) ,, F_1, F_2.

Let these be denoted by $K^{(1)}$, $K^{(2)}$, $K^{(3)}$ respectively.

The System $K^{(1)}$.

We enumerate these in ascending grade.

(i) Grade 1 has six forms

$$f,\ f',\ (Ap)^2,\ (Bp)^2,\ u_\alpha^2,\ u_\beta^2.$$

(ii) Grade 2 has nine forms

$$\begin{pmatrix} i & \omega \\ \xi & j \end{pmatrix},$$

i, j being 1, 2, 3 in all their arrangements; and nine forms

$$\begin{pmatrix} i \\ j \end{pmatrix}^2.$$

(iii) Grade 3 has eighteen forms

$$\begin{pmatrix} i & k \\ \xi & j & \xi \end{pmatrix}, \qquad \begin{pmatrix} \omega & j & \omega \\ i & & k \end{pmatrix},$$

where $i \neq k$, but i, j, k are all other possible permutations of 1, 2, 3.

Note that these forms are exactly the same if i, k are interchanged. No obvious reductions exist for any of these forms.

(iv) Grade 4 has two types

$$\begin{pmatrix} i & k & \omega \\ \xi & j & l \end{pmatrix} \text{ and } \begin{pmatrix} i & k & i \\ & j & l \end{pmatrix},$$

where i, j, k, l must contain a repetition.

(α) Let i, j, k, l include a repetition of 2,

(β) ,, ,, ,, 1 or 3.

(α).

29. We may have

$$\begin{pmatrix} i & 2 & \omega \\ \xi & j & 2 \end{pmatrix}, \quad \begin{pmatrix} i & 2 & \omega \\ \xi & 2 & j \end{pmatrix}, \quad \begin{pmatrix} 2 & i & \omega \\ \xi & 2 & j \end{pmatrix}, \quad \begin{pmatrix} 2 & j & \omega \\ \xi & i & 2 \end{pmatrix},$$

where i, j denote 1, 3.

The second and fourth of these reduce by formula (A), if $i \neq j$. We must omit these cases.

Proof.—The chain $\ldots 1_2 2_3 \ldots = 1_2 \,.\, 2_2 \,.\, 2_3$ (in full)

$$\equiv 0 \text{ (modd } 2_2 \times 2_2,\ 2_2 \times 2_2') \text{ [by (A)]}.$$

(β).

Besides $\begin{pmatrix} i & j & \omega \\ \xi & j & i \end{pmatrix}$, where 2 does not appear, we may have

$$\begin{pmatrix} 2 & i & \omega \\ \xi & i & j \end{pmatrix}, \quad \begin{pmatrix} 2 & j & \omega \\ \xi & i & j \end{pmatrix}, \quad \begin{pmatrix} i & 2 & \omega \\ \xi & i & j \end{pmatrix}, \quad \begin{pmatrix} j & 2 & \omega \\ \xi & i & j \end{pmatrix},$$

and four similar forms with 2 as a suffix.

By squaring formulæ (C) and (D) we reduce the second of these four forms. For example,

$$F_1^2 \begin{pmatrix} 3 \\ \xi \end{pmatrix}^2 \equiv \begin{pmatrix} 3 \\ 3 \end{pmatrix}^2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}^2 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ \xi \end{pmatrix}^2 \begin{pmatrix} \omega \\ 3 \end{pmatrix}^2 - 2 \begin{pmatrix} 2 & 3 & \omega \\ \xi & 1 & 3 \end{pmatrix}.$$

Again, the third form

$$\begin{pmatrix} i & 2 & \omega \\ \xi & i & j \end{pmatrix} = i_\xi \,.\, i_i \,.\, 2_i \,.\, 2_j \,.\, \omega_j \equiv i_\xi \,.\, 2_i \,.\, \omega_j \,(\pm F_1 \,.\, i_\xi \pm i_j \,.\, 2_\xi \,.\, {}_i\omega),$$

by (C) or (D), and each term has a reducing factor $\begin{pmatrix} i \\ \xi \end{pmatrix}^2$ or $\begin{pmatrix} i & \omega \\ \xi & j \end{pmatrix}$.

The first and fourth forms, however, do not reduce.

The second type, for Grade 4, viz. $\begin{pmatrix} i & k & i \\ j & & l \end{pmatrix}$, is unaltered by interchanging i and k or j and l, as is seen by writing it in full.

There are seven irreducible forms

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 1 & & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 1 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 2 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 3 \\ 1 & & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 3 \\ 2 & & 3 \end{pmatrix},$$

omitting the form $\begin{pmatrix} i & 2 & i \\ 2 & & j \end{pmatrix}$, which reduces by formula (A).

Grade 5.

30. There are two sets of correlative forms

$$\begin{pmatrix} i & j & k \\ \xi & l & m & \xi \end{pmatrix} \text{ and } \begin{pmatrix} \omega & l & m & \omega \\ i & j & k \end{pmatrix},$$

where i, j, k are 1, 2, 3 in some order.

(α) First take forms with 2_2 as a factor, and let i, j denote 1, 3.

There are two sorts, $\begin{pmatrix} 2 & \cdots \\ \xi & 2 & & \xi \end{pmatrix}$ and $\begin{pmatrix} i & 2 & \cdots \\ \xi & 2 & & \xi \end{pmatrix}$, since the chain may be written backwards or forwards, $\begin{pmatrix} i & j & k \\ \xi & l & m & \xi \end{pmatrix}$ being the same as

$$\begin{pmatrix} k & j & i \\ \xi & m & l & \xi \end{pmatrix}.$$

Thus we have the four types

$$\begin{pmatrix} 2 & i & j \\ \xi & 2 & i & \xi \end{pmatrix}, \quad \begin{pmatrix} i & 2 & j \\ \xi & 2 & j & \xi \end{pmatrix}, \quad \begin{pmatrix} 2 & i & j \\ \xi & 2 & j & \xi \end{pmatrix}, \quad \begin{pmatrix} i & 2 & j \\ \xi & 2 & i & \xi \end{pmatrix}.$$

By squaring each of the formulæ (E), (F), we reduce the first of these four types.

The second reduces by formula (A).

The other two lead to four irreducible forms.

(β) Next take forms with no factor 2_2.

With 2 in the middle of the top row we may have

$$\begin{pmatrix} 1 & 2 & 3 \\ \xi & 3 & 1 & \xi \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ \xi & 1 & 3 & \xi \end{pmatrix},$$

of which the latter is reducible by formula (D), as it contains the combination $1_\xi \,.\, 1_1 \,.\, 2_3$.

With 2 at the end of the top row, we may have

$$\begin{pmatrix} 2\,i\,j \\ \xi\;i\;2\;\xi \end{pmatrix},\quad \begin{pmatrix} 2\,j\,i \\ \xi\;i\;2\;\xi \end{pmatrix},\quad \begin{pmatrix} 2\,i\,j \\ \xi\;i\;j\;\xi \end{pmatrix},\quad \begin{pmatrix} 2\,j\,i \\ \xi\;i\;j\;\xi \end{pmatrix},$$

The first reduces by (C) or (D), as it contains $i_i \,.\, j_2$ and ${}_i 2_\xi$. The second, containing, $2_i \,.\, j_2$ reduces by (A). The third, containing $2_i \,.\, j_j$ with j_ξ, reduces by (C) or (D). The fourth, representing two forms, involves the relation

$$\begin{pmatrix} 2\;3\;1 \\ \xi\;1\;3\;\xi \end{pmatrix} + \begin{pmatrix} 2\;1\;3 \\ \xi\;3\;1\;\xi \end{pmatrix} \equiv 0,$$

which is proved by multiplying formula (L) by $\xi^2 \,.\, 1^3 \,.\, 3^1$.

31. Summing up, Grade 5 has six forms ξ,

$$\begin{pmatrix} 2\;i\;j \\ \xi\;2\;j\;\xi \end{pmatrix},\quad \begin{pmatrix} i\;2\;j \\ \xi\;2\;i\;\xi \end{pmatrix},\quad \begin{pmatrix} 1\;2\;3 \\ \xi\;3\;1\;\xi \end{pmatrix},$$

and *one* of

$$\begin{pmatrix} 2\;i\;j \\ \xi\;j\;i\;\xi \end{pmatrix}.$$

Together with six ω forms, this makes twelve forms in all.

Grade 6.

32. There are two kinds

$$\text{I}\quad \begin{pmatrix} i\;j\;k\;\omega \\ \xi\;i_1\;j_1\;k_1 \end{pmatrix},$$

$$\text{II}\quad \begin{pmatrix} i\;j\;k\;i \\ i_1\;j_1\;k_1 \end{pmatrix},$$

i, j, k as well as i_1, j_1, k_1 being 1, 2, 3 in some order.

I $\begin{pmatrix} i\;j\;k\;\omega \\ \xi\;i_1\;j_1\;k_1 \end{pmatrix}$.—First let $j = 2$. If $i_1 = 2$, then, by formula (A), j_1 and i cannot differ, so that $j_1 = i$. This gives $\begin{pmatrix} i\;2\;k\;\omega \\ \xi\;2\;i\;k \end{pmatrix}$, which reduces at once by (G) or (H). Hence i_1 cannot be 2.

It follows that either $j_1 = 2$ or $k_1 = 2$.

If $j_1 = 2$, then by formula (A) i_1 and k cannot differ.

This gives the form $\begin{pmatrix} i & 2 & j & \omega \\ \xi & j & 2 & i \end{pmatrix}$. (9)

Again, if $k_1 = 2$, *i.e.* if we have

$$\begin{pmatrix} i & 2 & k & \omega \\ \xi & i_1 & j_1 & 2 \end{pmatrix},$$

then $j_1 = k$ by formula (A). This gives

$$\begin{pmatrix} i & 2 & k & \omega \\ \xi & i & k & 2 \end{pmatrix},$$

which reduces by (G) or (H).

This exhausts the case when $j = 2$. Correlatively it exhausts the case when $j_1 = 2$. Take now the remaining cases, when 2 is found among i, k as well as i_1, k_1:

(*a*) $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & j_1 & k_1 \end{pmatrix}$, (*b*) $\begin{pmatrix} 2 & j & k & \omega \\ \xi & i_1 & j_1 & 2 \end{pmatrix}$,

(*c*) $\begin{pmatrix} i & j & 2 & \omega \\ \xi & 2 & j_1 & k_1 \end{pmatrix}$, (*d*) $\begin{pmatrix} i & j & 2 & \omega \\ \xi & i_1 & j_1 & 2 \end{pmatrix}$.

In these (*a*) and (*d*) are correlative cases.

(*a*) If $j = j_1$, then $k = k_1$, and the form reduces, as it contains $j_2 \,.\, k_k \,.\, {}_k\omega$, by the first of formulæ (C) or (D).

This leaves the case $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & k & j \end{pmatrix}$. Similarly for (*d*).

(*b*) If $j = j_1$, the form reduces by formula (G) or (H), as it contains $2_i \,.\, j_j \,.\, i_2$. Similarly for (*c*).

But if $j \neq j_1$, then $i_1 \neq k$, and the form reduces by formula (A), as it contains 2_i, k_2 $(i \neq k)$. Similarly for (*c*).

Altogether we have then $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & k & j \end{pmatrix}$ and $\begin{pmatrix} j & k & 2 & \omega \\ \xi & k & j & 2 \end{pmatrix}$. But multiplying formula (L) by $1_3 \,.\, 3_1 \,.\, 2_2 \,.\, {}_2\omega$, we get

$$\begin{pmatrix} 1 & 3 & 2 & \omega \\ \xi & 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 2 & \omega \\ \xi & 1 & 3 & 2 \end{pmatrix} \equiv 0.$$

Similarly for $\begin{pmatrix} 2 & j & k & \omega \\ \xi & 2 & k & j \end{pmatrix}$.

Again $\begin{pmatrix} 2 & 1 & 3 & \omega \\ \xi & 2 & 3 & 1 \end{pmatrix} = 2_2 \,.\, 3_3 \,.\, 3_1 \,.\, 1_2 \,\{1_1 \,.\, 2_3 - F_1 \,.\, 1_\xi\}$, by (D_2).

The first term reduces by applying formula (A) to $1_2 . 2_3$. The second term contains $F_1 . 1_2$. Using (E_2), this

$$\equiv \begin{pmatrix} 1 & 3 & \omega \\ \xi & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}^2 - \begin{pmatrix} 1 & 3 & 2 & \omega \\ \xi & 3 & 1 & 2 \end{pmatrix}.$$

Hence forms (a), (d) are all equivalent. We retain one of them.

33. II $\begin{pmatrix} i & j & k & i \\ i_1 & j_1 & k_1 & \end{pmatrix}$.—Without loss of generality we may consider only

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ i & j & k & \end{pmatrix}.$$

(a) If $j = 2$, then i cannot be 1, by formula (A); nor can $i = 3$, for then $k = 1$, and the form contains $1_1 . 3_2 . 2_3$, which reduces by (H).

(b) If $i = 2$, then $j \neq 3$ by (A); nor can $j = 1$, by (G).

(c) If $k = 2$, then $j \neq 3$ by (H); nor can $i = 3$, by (A).

So all the forms reduce.

34. Summing up, Grade 6 has three forms

$$\begin{pmatrix} 1 & 2 & 3 & \omega \\ \xi & 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 1 & \omega \\ \xi & 1 & 2 & 3 \end{pmatrix},$$

and *one* of

$$\begin{pmatrix} 2 & i & j & \omega \\ \xi & 2 & j & i \end{pmatrix}.$$

There are no irreducible forms of higher grade, as all the upper or lower symbols must differ, so that three is the maximum number of each.

The System $K^{(2)}$.

35. This consists of chains involving $2_2'$.

By formulæ (A), (B), the combination $2_2' . i_j$ is reducible if i, j denote 1, 1 or 1, 3 or 3, 1 or 3, 3; for the formulæ express these factors in terms of forms not involving $2_2'$ at all.

It follows that the only possible chains involve not more than two symbols i, j.

Again, the chain $\begin{pmatrix} i & 2' \\ 2 & j \end{pmatrix}$ is reducible by (A) if $i \neq j$. Hence we re-

tain the following nine forms :—

$$\begin{pmatrix} 2' & \omega \\ \xi & 2 \end{pmatrix}, \quad \begin{pmatrix} 2' \\ 2 \end{pmatrix}^2, \quad \begin{pmatrix} 2' & 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2' & i \\ \xi & 2\ \xi \end{pmatrix}, \quad \begin{pmatrix} \omega & 2' & \omega \\ 2 & & i \end{pmatrix}, \quad \begin{pmatrix} i & 2' & \omega \\ \xi & 2 & i \end{pmatrix}.$$

$$i = 1, 3.$$

The System $K^{(3)}$.

36. This system, involving factors F_1 and F_2, consists of two parallel sets: $J^{(4)}$ involving F_1 alone and $J^{(5)}$ involving F_2 alone. For, by formula (J), the product F_1F_2 is reducible.

We consider then forms $P = F_1M$, where M consists only of chains and tags.

Formulæ (C) to (K) shew that we need not consider terms with the factors

$$3_\xi,\ 1_\xi,\ 1_2,\ 3_2,\ 2'_2. \qquad (10)$$

There are three cases, besides the isolated form F_1^2, viz.

$$\text{I} \quad F_1 \begin{pmatrix} \\ 1 \end{pmatrix} \begin{pmatrix} \\ 3 \end{pmatrix}^{(2)}, \qquad \text{II} \quad F_1 \begin{pmatrix} \\ 1,\ 3 \end{pmatrix}^{(2)}, \qquad \text{III} \quad F_1 \begin{pmatrix} & 2 \\ i, & \end{pmatrix} \begin{pmatrix} \\ j \end{pmatrix},$$

where i, j are 1, 3.

I

$$F_1 \begin{pmatrix} \\ 1 \end{pmatrix} \begin{pmatrix} \\ 3 \end{pmatrix}^{(2)}.$$

37. The presence of (2) implies that in the other tags no upper link is 2. Likewise no suffix can be 1 or 3.

Thus $$F_1 \begin{pmatrix} i \dots \\ 1 \end{pmatrix} \begin{pmatrix} j \dots \\ 3 \end{pmatrix} \begin{pmatrix} 2 \dots \\ k \end{pmatrix}$$

can only be $$F_1 \cdot {}_1\omega \cdot {}_3\omega \cdot 2_\xi, \quad F_1 \cdot {}_1\omega \cdot {}_3\omega \begin{pmatrix} 2 & \omega \\ & 2 \end{pmatrix},$$

having regard to the list (10) given above.

II

$$F_1 \begin{pmatrix} \\ 1,\ 3 \end{pmatrix}^{(2)}.$$

38. The chain is $\begin{pmatrix} i \\ 1\ 3 \end{pmatrix}$ at most. For $\begin{pmatrix} i\ k \dots \\ 1\ j \quad\quad 3 \end{pmatrix}$ implies $j = 2$ and one of $i, k = 1$ or 3, and so is reducible. So we have three cases, $i = 1$, 2, or 3.

Case 1. $F_1\begin{pmatrix}1\\1\ 3\end{pmatrix}^{(2)}$.

The tag (2) may be 2_ξ,

$$\begin{pmatrix}2\ \omega\\ i\end{pmatrix},\quad i=1 \text{ or } 2 \text{ or } 3;\ \text{or } \begin{pmatrix}2\ j\ \cdots\\ i\end{pmatrix}:$$

here $j \neq 2$, so take $j = 1$ or 3.

If $j = 1$, then i cannot be 1 or 3 because of $\begin{pmatrix}1\\1\ 3\end{pmatrix}$. If $j = 1$, then i cannot be 2, since j_i would reduce by formula (E). Thus j cannot be 2 or 1. If however $j = 3$, then j_ξ is inadmissible. So the tag must contain another link k and be $\begin{pmatrix}2\ 3\ \cdots\\ i\ k\end{pmatrix}$. Here $i, k \neq 2$, since 3_2 is a reducing factor. Thus $i, k = 1, 3$; and the form is again reducible, because of the chain $\begin{pmatrix}1, 3\end{pmatrix}$ repeated. We are left with the above two types only. Of these, $F_1\begin{pmatrix}i\\1\ 3\end{pmatrix}\begin{pmatrix}2\ \omega\\ j\end{pmatrix}$ reduces if i, j are 1, 3 in either order. For this contains $a_i . 2_j . F_1 \equiv F_1^2 . j_\xi$ and reducible terms, by formulæ (C), (D).

Case 2. $F_1\begin{pmatrix}2\\1\ 3\end{pmatrix}^{(2)}$.

The tag (2) may be 2_ξ,

$2_2\omega$,

$\begin{pmatrix}2\ j\ \cdots\\ i\end{pmatrix}$;

here i can only be 2, because of the factors 2_1, 2_3 already in $\begin{pmatrix}2\\1\ 3\end{pmatrix}$. Hence j cannot be 1 or 3, because j_i would then reduce the form. Thus no form exists.

Case 3.

This is parallel with Case 1.

III.

$$F_1\begin{pmatrix} \cdots & 2 \\ i & \end{pmatrix}\begin{pmatrix} j \end{pmatrix} \quad (i, j \text{ being } 1, 3).$$

39. If the upper link 2 appears in $\begin{pmatrix} i \end{pmatrix}$, this becomes a form already discussed, having the chain $\begin{pmatrix} i, & j \end{pmatrix}$. Hence the upper links of $\begin{pmatrix} j \end{pmatrix}$ are i, j. It follows that 2 cannot be a suffix, as i_2, j_2 are reducing factors. Nor can i, j be suffixes.

This leaves j^ω, $j^k\xi$ ($k = 1$ or 3) as possible. Of these the latter reduces, because of k_ξ. Hence j^ω alone remains.

The possible forms are $F_1 . i^2 . j^\omega$,

$$F_1\begin{pmatrix} q & \cdots & 2 \\ i & r & \end{pmatrix} j^\omega ;$$

here $r \neq i, j$, Case II. But if $r = 2$, then $q \neq i, j$. Nor can $q = 2$. Hence the form always reduces.

40. In Table B overleaf these results are collected. But to make the system more accessible, the 125 forms given below are sorted into their orders in x, p, u, translated back into their original symbolic notation. Reducible terms in the expansion of F_1^2, and the like, are omitted. These sets are given at the beginning in Table A.

VI.

41. TABLE B.

Arranged in Groups to correspond with those of Gordan.

References.

The 21 *Forms* $J^{(1)}$.

$a_x^2,\ b_x^2,\ (Ap)^2,\ (Bp)^2,\ u_\alpha^2,\ u_\beta^2,\ (abp)^2,\ (aBu)^2,\ (bAu)^2,$
$a_\beta^2,\ b_\alpha^2,\ (AB)^2,\ (AB)(AB)',\ (AB)'^2,\ (A\beta x)^2,\ (Bax)^2,$
$(a\beta p)^2,\ F_1^2,\ F_2^2,\ a_\alpha^2,\ b_\beta^2.$

The 7 *Forms* $J^{(2)}$.

§ 29. $\begin{pmatrix} i\ 2\ i \\ 2\ i \end{pmatrix},\ \begin{pmatrix} i\ j\ i \\ 2\ i \end{pmatrix},\ \begin{pmatrix} i\ 2\ i \\ i\ j \end{pmatrix},\ \begin{pmatrix} 1\ 3\ 1 \\ 1\ 3 \end{pmatrix};\quad i, j = 1, 3.$

The 73 *Forms* $J^{(3)}$.

$\begin{pmatrix} i\ \omega \\ \xi\ j \end{pmatrix},\ \begin{pmatrix} i\ k \\ \xi\ j\ \xi \end{pmatrix},\ \begin{pmatrix} \omega\ j\ \omega \\ i\ k \end{pmatrix};\quad i, j, k = 1, 2, 3.$

Grade 4, § 29. $\begin{pmatrix} i\ 2\ \omega \\ \xi\ j\ 2 \end{pmatrix},\ \begin{pmatrix} i\ 2\ \omega \\ \xi\ 2\ i \end{pmatrix},\ \begin{pmatrix} 2\ i\ \omega \\ \xi\ 2\ j \end{pmatrix},\ \begin{pmatrix} 2\ i\ \omega \\ \xi\ i\ 2 \end{pmatrix};\quad i, j = 1, 3.$

$\begin{pmatrix} i\ j\ \omega \\ \xi\ j\ i \end{pmatrix},$

§ 29. $\begin{pmatrix} 2\ i\ \omega \\ \xi\ i\ j \end{pmatrix},\ \begin{pmatrix} j\ 2\ \omega \\ \xi\ i\ j \end{pmatrix},\ \begin{pmatrix} j\ i\ \omega \\ \xi\ i\ 2 \end{pmatrix},\ \begin{pmatrix} j\ i\ \omega \\ \xi\ 2\ j \end{pmatrix}.$

Grade 5, § 31. $\begin{pmatrix} 2\ i\ j \\ \xi\ 2\ j\ \xi \end{pmatrix},\ \begin{pmatrix} i\ 2\ j \\ \xi\ 2\ i\ \xi \end{pmatrix},\ \begin{pmatrix} 1\ 2\ 3 \\ \xi\ 3\ 1\ \xi \end{pmatrix},\ \begin{pmatrix} 2\ i\ j \\ \xi\ j\ i\ \xi \end{pmatrix},$

and four forms similarly for ω.

Grade 6, § 34. $\begin{pmatrix} i\ 2\ j\ \omega \\ \xi\ j\ 2\ i \end{pmatrix},\ \begin{pmatrix} 2\ j\ i\ \omega \\ \xi\ 2\ i\ j \end{pmatrix}.$

§ 35. $\begin{pmatrix} 2'\ \omega \\ \xi\ 2 \end{pmatrix},\ \begin{pmatrix} 2'\ i \\ \xi\ 2\ \xi \end{pmatrix},\ \begin{pmatrix} \omega\ 2'\ \omega \\ 2\ i \end{pmatrix},\ \begin{pmatrix} i\ 2'\ \omega \\ \xi\ 2\ i \end{pmatrix}.$

The actual numbers of each kind are, respectively,

$$(9+9+9)+(4+2+4+2)+4+(2+2+2+2)+2\,(2+2+1+1)$$
$$+(2+1)+1+2+2+2=73.$$

The 12 *Forms* $J^{(4)}$.

§§ 37–39. $F_{1\cdot1}\omega_{\cdot3}\omega_{\cdot}2_\xi,\ F_{1\cdot1}\omega_{\cdot3}\omega\begin{pmatrix} 2\ \omega \\ 2 \end{pmatrix},\ F_1\begin{pmatrix} i \\ 1\ 3 \end{pmatrix}2_\xi,\ F_1\begin{pmatrix} i \\ 1\ 3 \end{pmatrix}\begin{pmatrix} 2\ \omega \\ i \end{pmatrix},$
$F\begin{pmatrix} i \\ 1\ 3 \end{pmatrix}\begin{pmatrix} 2\ \omega \\ 2 \end{pmatrix},\ F_1\begin{pmatrix} 2 \\ 1\ 3 \end{pmatrix}2_\xi,\ F_1\begin{pmatrix} 2 \\ 1\ 3 \end{pmatrix}\begin{pmatrix} 2\ \omega \\ 2 \end{pmatrix},\ F_{1\cdot1}2_{\cdot3}\omega,$
$F_{1\cdot3}2_{\cdot1}\omega;\quad i = 1 \text{ or } 3.$

The 12 *Forms* $J^{(5)}$ *are similar.*

THE NATURE OF A MOVING ELECTRIC CHARGE AND ITS LINES OF ELECTRIC FORCE

By H. Bateman.

[Read January 18th, 1916.—Enlarged May 18th, 1917, and May 9th, 1918.]

1. *Introduction.*

Recent developments in the theory of radiation point to the conclusion that the electromagnetic theory of radiation is at present incomplete, and, in his "Report on Radiation and the Quantum Theory", J. H. Jeans* suggests that Maxwell's equations for the propagation of electric waves may need to be modified in such a way that the elementary electric charge e and Planck's constant h will be involved explicitly. There is, of course, the alternative that e and h may appear as constants of integration in an important type of solution of Maxwell's equations, and the idea of a magneton has already been used with some success in electromagnetic theory.† It has been suggested by some that the atomic nucleus, by others that the electron is the ultimate magnetic particle.

The idea of a magnetic electron‡ which consists of a thin ring of electricity in rapid motion is gaining ground, and has already received some experimental support.§ The radius of the ring has indeed been estimated to be about 2×10^{-10} cm., or about 500 times the radius generally adopted for a spherical electron.

These new ideas concerning the form of the electron are very interesting, but they do not solve the fundamental problem relating to the

---

* *Proc. Phys. Soc.*, London (1914), p. 81.

† See, for instance, S. B. McLaren, *Brit. Assoc. Report*, Birmingham (1913), p. 391: W. M. Hicks, *Proc. Roy. Soc.* (A), Vol. 90 (1914), p. 356; Vol. 91 (1915), p. 273.

‡ A. L. Parson, "A Magneton Theory of the Structure of the Atom", *Smithsonian Publication*, 2371, Washington (1915).

§ A. H. Compton and O. Rognley, *Phys. Review*, Vol. 11 (1918), p. 132; A. H. Compton, *J. Washington Academy of Science*, Vol. 8 (1918), Jan. 4.

‖ A. H. Compton (*l.c.*). Parson estimated his magneton to have a radius of $1{\cdot}5\times10^{-9}$ cm. D. L. Webster, *Phys. Review*, Vol. 9 (1917), p. 484, estimates the mass to be $2/c^2$ times the electrostatic energy and gives a method of finding the thickness of the ring.

manner in which an electron produces disturbances which are radiated outwards. To throw light on this question it seems necessary to frame some reasonable hypothesis as to the nature of the æther, and to make a careful study of Maxwell's equations, for in spite of the objections which have been raised against them these equations still provide us with the best starting point, especially since they can be adapted to theories of both continuous and discontinuous media.

The idea of a discontinuous æther may very well be the key to some of our difficulties in the theory of radiation, in fact in his report Jeans seems inclined to favour the idea of discrete Faraday tubes which has been so strongly recommended by Sir Joseph Thomson on many occasions,* and used as a basis for his theory of valency.† There is much to be said in support of such a theory. It is well known that the mathematical theory of electricity was much enriched when Maxwell translated Faraday's ideas relating to lines of force into mathematical language, and when Sir Joseph Thomson continued the work by considering the case of moving lines of induction. In this way a vivid picture was obtained of the processes taking place in an electromagnetic field and light was represented geometrically as traverse oscillations travelling along the lines of force.

These researches do not, however, exhaust the subject. We need to know more about the nature of the Faraday tubes, their connection with electric charges, their form in varying fields, and so forth. We need to know something about the origin of the potential energy in an electrostatic field, for mathematicians have thought for a long time that all energy is ultimately kinetic, and the importance of this way of looking at the matter has been emphasized by Sir Joseph Thomson.‡ Some of his remarks are of special interest in connection with the theory developed in this paper; when considered in connection with his theory of moving Faraday tubes,§ they strongly suggest the conclusions that are indicated directly by certain solutions of Maxwell's equations.

In this paper we shall follow Sir Joseph Larmor‖ in regarding Maxwell's equations for the æther as the starting point for the construction of more elaborate theories, and shall adopt the idea that all electricity

* *Phil. Mag.* (5), Vol. 31 (1891), p. 149; Vol. 19 (1910), p. 301; Vol. 26 (1913), p. 792: *Proc. Camb. Phil. Soc.*, Oct. (1907).

† *The Corpuscular Theory of Matter*, London (1907); *Phil. Mag.*, Vol. 27 (1914), p. 757.

‡ *Applications of Dynamics to Physics and Chemistry* (1888), p. 15.

§ As, for instance, by D. N. Mallik, *Phil. Mag.*, Vol. 26 (1913), p. 144; *Optical Theories*, Camb. Univ. Press (1917).

‖ *Æther and Matter*, p. 77.

arises from electrical separation.* To elucidate this idea a special study will be made of certain solutions of Maxwell's equations which seem to be of an elementary nature. The principal result of the investigation is that there is a type of radiant field which seems to be more fundamental than the electromagnetic field of a point charge of electricity, inasmuch as the flow of energy is rectilinear, and there is no transformation of energy outside the singularities of the field. The field of an electric pole moving with a velocity less than the velocity of light may, moreover, be obtained as the limiting case of two superposed fields of the type just mentioned, and the flow of energy in the field is generally of a complicated character; so there seem to be some advantages in adopting the simpler type of field as fundamental.

In order to obtain a clearer conception of the processes taking place in an electromagnetic field, we shall consider some types of fields in which there is a volume distribution of electricity, and shall introduce the idea of electric storage, a quantity whose rate of change is the volume density of electricity. Some geometrical theorems are obtained concerning the motion of lines of force, and these suggest very strongly that the lines of force are built up from particles of electricity.

2. *Solution of Maxwell's Equations.*

If the vectors E and H denote the electric and magnetic intensities in an electromagnetic field, and the units are chosen in a suitable way, Maxwell's equations for the æther may be written in the form

$$\left.\begin{aligned} c\,\mathrm{rot}\,\mathrm{H} &= \frac{\partial \mathrm{E}}{\partial t}, \qquad \mathrm{div}\,\mathrm{E} = 0 \\ c\,\mathrm{rot}\,\mathrm{E} &= -\frac{\partial \mathrm{H}}{\partial t}, \qquad \mathrm{div}\,\mathrm{H} = 0 \end{aligned}\right\}, \tag{1}$$

where c denotes the velocity of light and is regarded as constant.

Writing $\mathrm{M} = \mathrm{H} + i\mathrm{E}$, we obtain the equations in the concise form used by Weber and Silberstein,

$$c\,\mathrm{rot}\,\mathrm{M} = -i\,\frac{\partial \mathrm{M}}{\partial t}, \qquad \mathrm{div}\,\mathrm{M} = 0. \tag{2}$$

A particular solution of these equations may be obtained by writing

$$\mathrm{M} = \mathrm{m}f(\alpha, \beta),$$

* Cf. Larmor, *Proc. London Math. Soc.*, Ser. 2, Vol. 13 (1913), p. 50.

where α and β are functions of x, y, z, t, and $f(\alpha, \beta)$ is an arbitrary function of these quantities. Substituting in the equations we find that they are satisfied for all forms of the function f if

$$c[\nabla\alpha, \mathrm{m}]+i\frac{\partial\alpha}{\partial t}\mathrm{m}=0, \qquad (\mathrm{m}, \nabla\alpha)=0,$$

$$c[\nabla\beta, \mathrm{m}]+i\frac{\partial\beta}{\partial t}\mathrm{m}=0, \qquad (\mathrm{m}, \nabla\beta)=0,$$

$$c\,\mathrm{rot}\,\mathrm{m}+i\frac{\partial\mathrm{m}}{\partial t}=0, \qquad \mathrm{div}\,\mathrm{m}=0,$$

where $\nabla\theta$ denotes the vector whose components are $\partial\theta/\partial x$, $\partial\theta/\partial y$, $\partial\theta/\partial z$, and Lorentz's notation is used for the vector and scalar products.

The first two sets of equations give

$$\mathrm{m}=\gamma c\,[\nabla\alpha, \nabla\beta]=i\gamma\left\{\frac{\partial\beta}{\partial t}\nabla\alpha-\frac{\partial\alpha}{\partial t}\nabla\beta\right\};$$

and, when these expressions are substituted in the last set of equations, a set of equations is obtained which implies that γ is a function of α and β.

Hence we obtain a particular solution of type

$$\mathbf{M}=c[\nabla\alpha, \nabla\beta]F(\alpha, \beta)=i\left\{\frac{\partial\beta}{\partial t}\nabla\alpha-\frac{\partial\alpha}{\partial t}\nabla\beta\right\}F(\alpha, \beta), \qquad (3)$$

provided the functions α and β are related in such a way that the last equation holds.

To solve the last equation we may use the method of Ampère* and take α, β, x, y as new independent variables. The analysis has been given elsewhere, the result being

$$z-ct=\phi+\theta(x+iy), \qquad z+ct=\psi-\frac{1}{\theta}(x-iy), \qquad (4)$$

where θ, ϕ, and ψ are arbitrary functions of α and β. It is worthy of note that an arbitrary function of θ is a solution of the wave-equation.

When α and β are defined by these equations, the vector $\mathbf{M}$ given by (3) specifies a singular radiant field in which there is a rectilinear flow of energy.† It is easy to see that $M^2=0$, so that the energy flows with the velocity of light, and there is no accumulation or loss of energy at an ordinary point of space. It can be proved that the above solution of

* See Forsyth's *Differential Equations*, p. 533.

† *Messenger of Mathematics*, May 1915.

Maxwell's equations is the most general solution which possesses this property;* it can also be proved that if there are two solutions of Maxwell's equations such that their vectors $\mathbf{M}$ and $\mathbf{M}'$ are connected by a relation of type $\mathbf{M}' = k\mathbf{M}$, where k is not a constant, then $M^2 = 0$ and the solution provided by either $\mathbf{M}$ or $\mathbf{M}'$ is of the above type.

A more general solution of Maxwell's equations may be obtained by superposing a number of particular solutions of the present type, and one of the objects of the present memoir is show how some important types of electromagnetic fields may be built up by superposition in this way. It should be mentioned that the field obtained by superposing an infinite number of singular radiant fields is not necessarily a field in which Maxwell's equations are satisfied, for the singular curves of the radiant fields may fill the whole of space. It will be of some interest to aim at a second object and to endeavour to determine some of the properties of a general type of field that can be built up from singular radiant fields.

3. *A Simple Type of Singular Radiant Field.*†

Let us write $\psi = ca$, $\phi = -ca$, $\theta = e^{\beta}$ in (4): we then obtain a simple solution of type (3) in which

$$a = t - \frac{r}{c}, \qquad \beta = \log\frac{z-r}{x+iy}, \qquad r^2 = x^2+y^2+z^2,$$

The three components of $\mathbf{M}$ are easily found to be

$$M_x = F(a, \beta)\left\{\frac{y}{r(r-z)} - \frac{iz}{r(x+iy)}\right\},$$

$$M_y = F(a, \beta)\left\{\frac{x}{r(z-r)} + \frac{z}{r(x+iy)}\right\},$$

$$M_z = iF(a, \beta)\frac{1}{r}.$$

Since $xM_x+yM_y+zM_z = 0$, the components of $\mathbf{E}$ and $\mathbf{H}$ satisfy similar equations, and so $\mathbf{E}$ and $\mathbf{H}$ are both at right angles to the radius from the origin. Poynting's vector is thus along the radius, and it is easy to see that the flow of energy is away from the origin. It should be noticed that

* This theorem and the following one are proved in a paper published in the *Messenger of Mathematics* in March 1918.

† Cf. O. Heaviside, *The Electrician*, Nov. 29 1901; *Electromagnetic Theory*, Vol. 3, p. 122.

on the axis of z there is a radical component of $\mathbf{E}$, but as the transverse components are now infinite there is no contradiction. The axis of z is a line of singularities and must be regarded as the locus of point charges of electricity which are moving away from the origin with the velocity of light; this at any rate is the correct interpretation when $F(\alpha, \beta)$ has certain forms. In some cases the moving singularities are magnetic poles instead of electric, while if $F(\alpha, \beta)$ is a many valued function it may be difficult to give a physical meaning to the singularities.

Let us now consider the simple case in which $F(\alpha, \beta)$ depends only on α. There are then no other singularities in the field except perhaps those of the function $F(\alpha)$. Let us suppose moreover that $F(\alpha, \beta)$ is zero except when α differs from α_0 by an infinitesimal amount: then $\mathbf{M}$ is infinite only at points on the axis of z which move according to the law $|r| = c(t-\alpha_0)$. If now we assume that these two points carry charges of electricity, it is easy to see that they must carry charges of opposite sign, for the surface integral of normal electric force over a sphere whose centre is at O is evidently zero if the radius of the sphere is so large that the moving charges have not reached it at the moment under consideration. It appears then that the singular radiant field arises from the breaking up at O of neutral particles into oppositely charged particles which fly away in opposite directions with the velocity of light.

To verify this let us write $F(\alpha, \beta) = f'(\alpha)$, where $f'(\alpha)$ is zero when $\alpha < t_0$ and when $\alpha > t_1$. Then at any instant subsequent to t_1 we can determine the amount of electricity which has been fired out from O in one direction by calculating the surface integral of the normal electric force over the xy plane. This surface integral is

$$\int_0^{2\pi}\int_0^{\infty} \frac{1}{r} f'\left(t-\frac{r}{c}\right) r\,d\theta\,dr = 2\pi c\,f(t) = 2\pi c\,f(t_1),$$

and so the amount of electricity which has travelled along the axis of z in the positive direction is $-2\pi c\,f(t_1)$, while the amount which has travelled in the negative direction is $2\pi c\,f(t_1)$.

We have assumed of course that $f(\alpha)$ is real, and it is easy to see that, since $H_z = 0$, there is no magnetic charge associated with the singularities. When $F(\alpha, \beta)$ is complex both electricity and magnetism occur. In the general case when $F(\alpha, \beta) = G(\alpha)$, where $G(\alpha)$ is real, the amount of electricity which has travelled along the axis of z in the negative direction up to time t is equal to

$$2\pi c \int_{-\infty}^{t} G(\alpha)\,d\alpha.$$

Let us now take F to be zero up to $t = t_0$ and constant but not zero

afterwards: then if we differentiate the expressions for the components of **M** with regard to z, we find that when F is constant

$$\frac{\partial}{\partial z} M_x = -\frac{iFx}{r^3}, \quad \frac{\partial}{\partial z} M_y = -\frac{iFy}{r^3}, \quad \frac{\partial}{\partial z} M_z = -\frac{iFz}{r^3}.$$

The differentiation of the field may be regarded as the limit of an operation in which two singular radiant fields issuing from two neighbouring sources O and O' on ths axis of z are superposed. The functions $G(\alpha)$ associated with the two sources are equal but of opposite signs, the magnitude of G being $F/\delta z$, where δz is the distance between O and O'. The above expressions indicate that, in the limit when δz is zero, the two superposed radiant fields are equivalent to the electrostatic field of a point charge $-F$ at O, but the equivalence is valid at time t only for points which lie within the sphere $r = c(t-t_0)$. Outside this sphere there is no field at all.

A similar result may be obtained by using Heaviside's expressions for the electric and magnetic forces. E and H are both equal in magnitude to F/ϖ, where ϖ is the distance of a point (x, y, z) from the axis of z. The lines of magnetic force are circles in planes perpendicular to the axis of z, the centres of the circles being on the axis of z. The lines of electric force are circles in planes through the axis of z, their centres being at the point O. The field is, of course, limited to a sphere of radius $c(t-t_0)$ with its centre at the origin.

Now when two fields of this type are superposed so that their singular lines overlap, it is evident that, if the functions F are equal but of opposite signs, the magnetic forces in the two fields will cancel one another at all points where both fields exist. The fragments of electricity fired out from O and O' will also annul one another except at points of the interval OO'. The two superposed fields are consequently equivalent to the electrostatic field of a uniformly charged rod OO', but this equivalence is valid only within the region common to two expanding spheres whose centres are at O and O'.

If the separation of electricity begins at O and is initiated at O' as soon as O' is struck by a fragment of electricity from O, we shall have a constant electric charge associated with the interval OO', and an equal and opposite charge spread over an interval of twice the length and travelling along the axis of z with the speed of light. This moving charge ought to be annulled in some way, and the simplest way of doing this is to introduce two other sources Q and Q' at suitable points on the axis of z. In order that a cancelling may take place, the electric charge associated with Q and Q' must be equal and opposite to that associated with

O and O', while the distance between Q and Q' must be the same as the distance between O and O'.

4. *The Electromagnetic Potentials for a Singular Radiant Field.*

It is evident from (3) that we may write

$$\mathbf{M} = i \operatorname{rot} \mathbf{L} = \frac{1}{c} \frac{\partial \mathbf{L}}{\partial t} + \nabla \Lambda,$$

where

$$\mathbf{L} = \mathbf{B} - i\mathbf{A} = -ic \int_{-\infty}^{\alpha} d\tau \nabla G(\tau, \sigma),$$

$$\Lambda = \Psi - i\Phi = \quad i \int_{-\infty}^{\alpha} d\tau \frac{\partial}{\partial t} G(\tau, \sigma).$$

The quantity σ is supposed to be a function of x, y, z, t and τ which is equal to β when $\tau = \alpha$, and G is supposed to be connected with F by the equation

$$\frac{\partial}{\partial \sigma} G(\tau, \sigma) = F(\tau, \sigma).$$

Let us now consider the simple case when the functions ϕ and ψ in equations (4), § 2, have the following forms

$$\phi(\alpha, \beta) = \xi(\alpha) - c\alpha - \theta(\alpha, \beta)[\xi(\alpha) + i\eta(\alpha)],$$

$$\psi(\alpha, \beta) = \xi(\alpha) + c\alpha + \frac{1}{\theta(\alpha, \beta)}[\xi(\alpha) - i\eta(\alpha)].$$

Then α is defined by the equation

$$[x - \xi(\alpha)]^2 + [y - \eta(\alpha)]^2 + [z - \zeta(\alpha)]^2 = c^2(t - \alpha)^2,$$

and β by the equation

$$\theta(\alpha, \beta) = \frac{z - \xi(\alpha) - c(t - \alpha)}{x - \xi(\alpha) + i[y - \eta(\alpha)]}.$$

Take

$$G(\alpha, \beta) = f'(\alpha) \log \frac{\lambda + \mu\theta}{\nu + \rho\theta},$$

where λ, μ, ν, and ρ, are arbitrary functions of α, and may be complex qnantities, then the corresponding radiant field has moving singular curves given by the equations

$$\lambda + \mu\theta = 0, \qquad \nu + \rho\theta = 0,$$

respectively. If we use bars to denote that $\bar{\lambda}$, $\bar{\theta}$, &c., are the complex

quantities conjugate to λ, θ, &c., we may write

$$\frac{\lambda+\mu\theta}{\nu+\rho\theta} = \frac{(\lambda+\mu\theta)(\bar\lambda+\bar\mu\bar\theta)}{(\nu+\rho\theta)(\bar\lambda+\bar\mu\bar\theta)} = \frac{l(x-\xi)+m(y-\eta)+n(z-\zeta)-c(t-\alpha)}{l_1(x-\xi)+m_1(y-\eta)+n_1(z-\zeta)-cp_1(t-\alpha)},$$

where

$$\lambda\bar\mu+\bar\lambda\mu = \kappa l, \quad i(\lambda\mu-\bar\lambda\mu) = \kappa m, \quad \mu\bar\mu-\lambda\bar\lambda = \kappa n, \quad \mu\bar\mu+\lambda\bar\lambda = \kappa,$$

$$\rho\bar\lambda+\nu\bar\mu = \kappa l_1, \quad i(\nu\bar\mu-\bar\lambda\rho) = \kappa m_1, \quad \rho\bar\mu-\nu\bar\lambda = \kappa n_1, \quad \rho\bar\mu+\nu\bar\lambda = \kappa p_1.$$

We may also write

$$\frac{\lambda+\mu\theta}{\nu+\rho\theta} = \frac{(\lambda+\mu\theta)(\bar\nu+\bar\rho\bar\theta)}{(\nu+\rho\theta)(\bar\nu+\bar\rho\bar\theta)} = \frac{1}{\omega}\,\frac{\bar l_1(x-\xi)+\bar m_1(y-\eta)+\bar n_1(z-\zeta)-c\bar p_1(t-\alpha)}{l_0(x-\xi)+m_0(y-\eta)+n_0(z-\zeta)-c(t-\alpha)},$$

where

$$\bar\rho\lambda+\bar\nu\mu = \kappa\bar l_1, \quad i(\lambda\bar\rho-\bar\nu\mu) = \kappa\bar m_1, \quad \bar\rho\mu-\bar\nu\lambda = \kappa\bar n_1, \quad \bar\rho\mu+\bar\nu\lambda = \kappa\bar p_1,$$

$$\nu\bar\rho+\bar\nu\rho = \omega\kappa l_0, \quad i(\nu\bar\rho-\bar\nu\rho) = \omega\kappa m_0, \quad \rho\bar\rho-\nu\bar\nu = \omega\kappa n_0, \quad \rho\bar\rho+\nu\bar\nu = \omega\kappa.$$

If $\mathfrak{s}$, $\mathfrak{s}_0$, $\mathfrak{s}_1$, $\bar{\mathfrak{s}}_1$ denote the vectors whose components are (l, m, n), (l_0, m_0, n_0), (l_1, m_1, n_1), $(\bar l_1, \bar m_1, \bar n_1)$ respectively, it is easy to see that

$$s^2 = 1, \quad s_0^2 = 1, \quad s_1^2 = p^2, \quad s_1^2 = \bar p_1^2, \quad (\mathfrak{s}\mathfrak{s}_1) = p_1, \quad (\mathfrak{s}\bar{\mathfrak{s}}_1) = \bar p_1,$$

$$(\mathfrak{s}_0\mathfrak{s}_1) = p_1, \quad (\mathfrak{s}_0\bar{\mathfrak{s}}_1) = \bar p_1.$$

Now let r denote the radius vector from the point $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ to the point (x, y, z). Then, if τ is an arbitrary real quantity, we shall obtain expressions analogous to the above with τ instead of α, if we write r instead of $c(t-\tau)$. Consequently we may write

$$\mathfrak{B}-i\mathfrak{A} = -ic\int_{-\infty}^{\alpha} f'(\tau)\,d\tau\nabla\log\frac{(\mathfrak{s}\mathfrak{r})-r}{(\mathfrak{s}_1\mathfrak{r})-p_1 r},$$

$$\Psi-i\Phi = \quad i\int_{-\infty}^{\alpha} f'(\tau)\,d\tau\frac{\partial}{\partial t}\log\frac{(\mathfrak{s}\mathfrak{r})-r}{(\mathfrak{s}_1\mathfrak{r})-p_1 r}.$$

Equating the real and imaginary parts and making use of the identity

$$\frac{(\mathfrak{s}\mathfrak{r})-r}{(\mathfrak{s}_1\mathfrak{r})-p_1 r} = \frac{1}{\omega}\,\frac{(\bar{\mathfrak{s}}_1\mathfrak{r})-\bar p_1 r}{(\mathfrak{s}_0\mathfrak{r})-r},$$

we find that we may write

$$2\mathrm{A} = c \int_{-\infty}^{\alpha} f'(\tau)\, d\tau \nabla \log \frac{(\mathrm{s}\mathrm{r})-r}{(\mathrm{s}_0\mathrm{r})-r},$$

$$2\Phi = -\int_{-\infty}^{\alpha} f'(\tau)\, d\tau \frac{\partial}{\partial t} \log \frac{(\mathrm{s}\mathrm{r})-r}{(\mathrm{s}_0\mathrm{r})-r},$$

$$2\mathrm{B} = ic \int_{-\infty}^{\alpha} f'(\tau)\, d\tau \nabla \log \frac{(\mathrm{s}_1\mathrm{r})-p_1 r}{(\bar{\mathrm{s}}_1\mathrm{r})-\bar{p}_1 r},$$

$$2\Psi = -i\int_{-\infty}^{\alpha} f'(\tau)\, d\tau \frac{\partial}{\partial t} \log \frac{(\mathrm{s}_1\mathrm{r})-p_1 r}{(\bar{\mathrm{s}}_1\mathrm{r})-\bar{p}_1 r}.$$

Now as far as the calculation of the components of E and H is concerned, we shall obtain the same result if we replace r by $c(t-\tau)$ in the above expressions. Consequently we shall adopt as our potentials

$$\mathrm{A} = \frac{c}{2} \int_{-\infty}^{\alpha} f'(\tau)\, d\tau \nabla \log \frac{(\mathrm{s}\mathrm{r})-c(t-\tau)}{(\mathrm{s}_0\mathrm{r})-c(t-\tau)},$$

$$\Phi = -\tfrac{1}{2}\int_{-\infty}^{\alpha} f'(\tau)\, d\tau \frac{\partial}{\partial t} \log \frac{(\mathrm{s}\mathrm{r})-c(t-\tau)}{(\mathrm{s}_0\mathrm{r})-c(t-\tau)},$$

$$\mathrm{B} = \frac{ic}{2} \int_{-\infty}^{\alpha} f'(\tau)\, d\tau \nabla \log \frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)}{(\bar{\mathrm{s}}_1\mathrm{r})-c\bar{p}_1(t-\tau)},$$

$$\Psi = -\frac{ic}{2}\int_{-\infty}^{\alpha} f'(\tau)\, d\tau \frac{\partial}{\partial t} \log \frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)}{(\bar{\mathrm{s}}_1\mathrm{r})-c\bar{p}_1(t-\tau)}.$$

These satisfy the equations

$$\operatorname{div}\mathrm{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0, \quad \operatorname{div}\mathrm{B} + \frac{1}{c}\frac{\partial\Psi}{\partial t} = 0,$$

and are solutions of the wave-equation, so that they are electromagnetic potentials in the ordinary sense.

From these potentials we find that

$$\mathrm{H} = [\nabla\alpha, \nabla\theta], \quad c\mathrm{E} = \frac{\partial\theta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\theta,$$

$$\mathrm{E} = [\nabla\alpha, \nabla\phi], \quad c\mathrm{H} = \frac{\partial\alpha}{\partial t}\nabla\phi - \frac{\partial\phi}{\partial t}\nabla\alpha,$$

where

$$\theta = \frac{c}{2} f'(\alpha) \log \frac{(\mathrm{s}\mathrm{r})-c(t-\alpha)}{(\mathrm{s}_0\mathrm{r})-c(t-\alpha)}, \qquad \phi = \frac{ic}{2} f'(\alpha) \log \frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\alpha)}{(\bar{\mathrm{s}}_1\mathrm{r})-c\bar{p}_1(t-\alpha)}.$$

In these expressions $\mathfrak{s}$, $\mathfrak{s}_0$, $\mathfrak{s}_1$, and $\bar{\mathfrak{s}}_1$ are functions of a, while $\mathfrak{r}$ now denotes the radius vector from the point $[\xi(a),\ \eta(a),\ \zeta(a)]$ to x, y, z.

The singular radiant field specified by the above expressions has two singular curves, each of which is the locus of points travelling along straight lines with the velocity of light. These moving points may be supposed to be projected from the different positions of the primary singularity $[\xi(a), \eta(a), \zeta(a)]$ in directions specified by the unit vectors $\mathfrak{s}(a)$ and $\mathfrak{s}_0(a)$ respectively.

The lines of electric force $\alpha = \text{const.}$, $\phi = \text{const.}$ form at any instant a system of circles meeting the two singular curves of the radiant field. The lines of magnetic force $\alpha = \text{const.}$, $\theta = \text{const.}$ also form a system of circles. The lines of electric and magnetic force which lie on a sphere, with its centre at some position of the primary singularity, form two orthogonal systems of circles cut out by planes through two lines which are polar lines with respect to the sphere.

From the expressions for E it is easy to prove that, if

$$\epsilon = \frac{\partial a}{\partial t} f'(a),$$

the electric force is the resultant of a force ϵ/r_2 directed away from the position at time t of one of the singularities projected from the primary singularity at time a, and a force ϵ/r_1 towards the position at time t of the other singularity which was projected from the primary singularity at time a.

The projected singular points evidently have electric charges associated with them, and as before we may imagine neutral particles to be continually breaking up at the primary singularity into oppositely charged fragments which fly away with the velocity of light. To verify that this is the correct interpretation of our singular radiant field we may proceed as before, but the work is laborious. The following method will, however, give us a partial verification.

[*Added May 9th*, 1918.—

Let us consider the electromagnetic field specified by the potentials

$$\mathrm{A} = \frac{c}{2} \int_{-\infty}^{a} f'(\tau) d\tau \nabla \log[(\mathfrak{sr}) - c(t-\tau)],$$

$$\Phi = -\tfrac{1}{2} \int_{-\infty}^{a} f'(\tau) d\tau \frac{\partial}{\partial t} \log[(\mathfrak{sr}) - c(t-\tau)].$$

These satisfy the wave equation but are connected by the relation

$$\operatorname{div}\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=\Omega,$$

where

$$\Omega=\frac{cf'(\alpha)}{2N},$$

and $N=\xi'(\alpha)[x-\xi(\alpha)]+\eta'(\alpha)[y-\eta(\alpha]+\zeta'(\alpha)[z-\xi(\alpha)]-c^2(t-\alpha).$

The electric and magnetic intensities derived from $\mathbf{A}$ and Φ satisfy the equations

$$c\operatorname{rot}\mathbf{H}=\frac{\partial\mathbf{E}}{\partial t}+\rho\mathbf{v},\qquad \operatorname{div}\mathbf{E}=\rho,$$

$$c\operatorname{rot}\mathbf{E}=-\frac{\partial\mathbf{H}}{\partial t},\qquad \operatorname{div}\mathbf{H}=0,$$

where

$$\rho\mathbf{v}=c\nabla\Omega,\qquad \rho=-\frac{1}{c}\frac{\partial\Omega}{\partial t}.$$

The volume density of electricity provides a total charge which at time t is equal to

$$-\frac{1}{c}\iiint\frac{\partial\Omega}{\partial t}\,dx\,dy\,dz.$$

Now we may write

$$dx\,dy\,dz=cN(t-\alpha)\sin\theta\,d\theta\,d\phi\,d\alpha,$$

$$x=\xi+c(t-\alpha)\sin\theta\cos\phi,\qquad y=\eta+c(t-\alpha)\sin\theta\sin\phi,$$

$$z=\zeta+c(t-\alpha)\cos\theta,$$

and the total charge becomes

$$e=\tfrac{1}{2}\int_0^{\pi}\int_0^{2\pi}\int_{-\infty}^{t}\left\{-\frac{f''(\alpha)}{N^2}c^2(t-\alpha)+\frac{f'(\alpha)}{N^3}Kc^2(t-\alpha)+\frac{c^2f'(\alpha)}{N^2}\right\}$$

$$\times cN(t-\alpha)\sin\theta\,d\alpha\,d\phi\,d\theta,$$

where $K=\xi''(x-\xi)+\eta''(y-\eta)+\zeta''(z-\zeta)+c^2-\xi'^2-\eta'^2-\zeta'^2.$

Writing $J=\sin\theta\cos\phi\,\xi'(\alpha)+\sin\theta\sin\phi\,\eta'(\alpha)+\cos\theta\,\zeta'(\alpha)-c,$

we find that

$$e=\tfrac{1}{2}\int_0^{\pi}\int_0^{2\pi}\int_{-\infty}^{t}\left[c(c^2-\xi'^2-\eta'^2-\zeta'^2)\frac{f'(\alpha)}{J^2}-c^2\frac{d}{d\alpha}\left\{\frac{f'(\alpha)}{J}(t-\alpha)\right\}\right]$$

$$\times\sin\theta\,d\alpha\,d\phi\,d\theta.$$

If $f(-\infty) = 0$, the first term gives us

$$2\pi c \int_{-\infty}^{t} f'(\alpha)\, d\alpha = 2\pi c\, f(t),$$

while the second term gives us nothing. The total charge which has been created in a concentrated form as point singularities up to time t is thus $-2\pi c\, f(t)$.

The quantity $-1/c\,\Omega$, whose time derivative is equal to ρ, will be called the *electric storage* and denoted by S. The total storage between the spheres $\alpha - d\alpha$ and α is

$$-2\pi c\, f'(\alpha)(t-\alpha)\, d\alpha,$$

and this is equal to the electric storage associated with the element $-2\pi c\, f'(\alpha)\, d\alpha$ of concentrated charge, provided we define the storage as the product of the charge and the interval of time since it was created.

It should be noticed that the electric and magnetic intensities may be written in the form

$$\mathbf{E} = -\Omega \frac{\mathbf{r} - r\mathbf{s}}{(\mathbf{rs}) - r},$$

$$\mathbf{H} = \Omega \frac{[\mathbf{rs}]}{(\mathbf{rs}) - r}.$$

These expressions indicate that $E^2 - H^2 = \Omega^2$, that $(\mathbf{Es}) = \Omega$, that $(\mathbf{Er}) = -r\Omega$, and that $(\mathbf{EH}) = 0$.

Since $\mathbf{r}$ denotes the radius vector from $\xi(\alpha)$, $\eta(\alpha)$, $\zeta(\alpha)$ to x, y, z, and $\mathbf{s}$ is the unit vector which gives the direction in which the concentrated charge $-2\pi c\, f'(\alpha)\, d\alpha$ is fired out, it follows that $\mathbf{E}$ is in the direction of the radius joining x, y, z to the position of this concentrated charge at time t. This radius moves so as to remain parallel to itself as t varies, and the quantities which multiply Ω in the expressions for the components of $\mathbf{E}$ and $\mathbf{H}$ may be regarded as the six direction ratios of this moving line.*

The vectors $\mathbf{E}$ and $\mathbf{H}$ may be supposed to indicate the rate at which electric storage is being displaced away from the concentrated charge. It may be noticed in this connection that

$$\mathbf{E} + \frac{1}{r}[\mathbf{rH}] = \frac{\mathbf{r}}{r}\Omega, \qquad \mathbf{E} + [\mathbf{sH}] = -\mathbf{s}\Omega.$$

* When x, y, z and ict are interpreted as the coordinates of a point in a space of four dimensions, the moving line is represented by a plane and the six direction ratios are the direction cosines of this plane.

Now $\Omega = -cS$, so that the above vectors both represent rates at which storage is being displaced and may be used to define the vectors E and H.

The electric intensity is easily seen to be $\frac{2\Omega c(t-\alpha)}{r_1}$. Now we may write

$$\frac{\partial \alpha}{\partial t} = -\frac{c^2(t-\alpha)}{N};$$

hence the electric intensity may be expressed in the simpler form

$$E = -\frac{1}{r_1}\frac{\partial \alpha}{\partial t} f'(\alpha) = -\frac{1}{r_1}\frac{\partial}{\partial t} f(\alpha).$$

Our previous statement regarding the electric intensity of the field in which concentrated charges are fired in two directions is now easily verified.

The value of ρ and $\rho \mathbf{v}$ derived from the function Ω are such that v is not generally equal to c the velocity of light. The reason is that when the original electricity which is created is moving outwards, a secondary creation of electricity takes place in the region traversed, and some of this new electricity converges towards a new position of the primary singularity and tends to produce a steady state in which electricity of a concentrated type is continually formed. This phenomenon seems to be connected in some way with the permanence of electric charges, and is illustrated by a geometrical theorem concerning lines of force which will be proved presently.]

5. *The Interference of Singular Radiant Fields and the Cancelling of Singularities.*

Let us now consider two singular radiant fields whose singular curves overlap. Let O and O_1 be the primary singularities: then these must move in such a manner that electricity fired from O in one direction hits O_1, and electricity fired from O_1 in one direction hits O. The other directions in which electricity is fired from O and O_1 must coincide with the directions of motion of the particles of electricity which hit O and O_1 respectively.

Let the coordinates of O at time τ be $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$, then if ϵ is the time which one particle of electricity takes to go from O to O_1, the coordinates of O_1 at time $\tau+\epsilon$ are

$$\xi(\tau)+c\epsilon\, l(\tau), \quad \eta(\tau)+c\epsilon\, m(\tau), \quad \zeta(\tau)+c\epsilon\, n(\tau),$$

and it should be noticed that

$$l(x-\xi-c\epsilon l)+m(y-\eta-c\epsilon m)+n(z-\zeta-c\epsilon n)-c(t-\tau-\epsilon)$$
$$= l(x-\xi)+m(y-\eta)+n(z-\zeta)-c(t-\tau). \quad (1)$$

Again, if ϵ_1 is the duration of the journey from O_1 to O of the fragment of electricity which hits O at time τ, the coordinates of O_1 at time $\tau-\epsilon_1$ will be

$$\xi(\tau)-c\epsilon_1 l_0(\tau), \qquad \eta(\tau)-c\epsilon_1 m_0(\tau), \qquad \zeta(\tau)-c\epsilon_1 n_0(\tau),$$

and it should be noticed that

$$l_0(x-\xi+c\epsilon_1 l_0)+m_0(y-\eta+c\epsilon_1 m_0)+n_0(z-\zeta+c\epsilon_1 n_0)-c(t-\tau+\epsilon_1)$$
$$= l_0(x-\xi)+m_0(y-\eta)+n_0(z-\zeta)-c(t-\tau). \quad (2)$$

Now let two functions $f(\tau)$ and $\phi(\tau)$ be chosen so that*

$$f(\tau)-\phi(\tau-\epsilon_1) = T, \qquad \phi(\tau+\epsilon)-f(\tau) = T,$$

where T is a constant; then if we write $\tau-\epsilon_1 = \tau_0$, $\tau+\epsilon = \tau_1$, we find on differentiation that

$$f'(\tau)\,d\tau = \phi'(\tau_0)\,d\tau_0, \qquad \phi'(\tau_1)\,d\tau_1 = f'(\tau)\,d\tau. \quad (3)$$

Now let α_1 be the time at which a light-particle must start from O_1 in order to reach the point (x, y, z) at time t, and let α' be the time at which a light-particle must start from O in order to reach O_1 at time α_1, also let α'' be the time at which a light-particle reaches O when fired from O_1 at time α_1.

The electromagnetic potentials for a singular radiant field in which O_1 is the primary singularity and in which the singular curves partly coincide with those of the singular radiant field with O as primary singularity, can on account of relations (1), (2), and (3) be thrown into the forms

$$\mathbf{A} = \frac{c}{2}\int_{-\infty'}^{\alpha'} f'(\tau)\,d\tau\nabla\log[(\mathfrak{s}\mathfrak{r})-c(t-\tau)]$$
$$-\frac{c}{2}\int_{-\infty''}^{\alpha''} f'(\tau)\,d\tau\nabla\log[(\mathfrak{s}_0\mathfrak{r})-c(t-\tau)],$$

$$\Phi = -\tfrac{1}{2}\int_{-\infty'}^{\alpha'} f'(\tau)\,d\tau\frac{\partial}{\partial t}\log[(\mathfrak{s}\mathfrak{r})-c(t-\tau)]$$
$$+\tfrac{1}{2}\int_{-\infty''}^{\alpha''} f'(\tau)\,d\tau\frac{\partial}{\partial t}\log[(\mathfrak{s}_0\mathfrak{r})-c(t-\tau)],$$

* H. Bateman, *Messenger of Mathematics*, Vol. 14 (1915), p. 97.

where ∞' is not the same as ∞'' because the change of variables is not the same in the two cases when we pass from times associated with O_1 to times associated with O. If we suppose that electricity is first fired from O to O_1, we may write $\infty' = \infty$, $\infty' > \infty''$,

$$f(-\infty'')-f(-\infty') = 2T.$$

When we subtract the two singular radiant fields the portions of the above integrals that are associated with the range of integration from $-\infty'$ to $-\infty''$ may be neglected, and we obtain a field specified by the electromagnetic potentials

$$\mathfrak{a} = -\frac{c}{2}\int_a^{a'} f'(\tau)\,d\tau\nabla\,\log[(\mathfrak{sr})-c(t-\tau)]-\frac{c}{2}\int_a^{a''} f'(\tau)\,d\tau\nabla\log[(\mathfrak{s}_0\mathfrak{r})-c(t-\tau)],$$

$$\phi = -\tfrac{1}{2}\int_a^{a'} f'(\tau)d\tau\frac{\partial}{\partial t}\log[(\mathfrak{sr})-c(t-\tau)]+\tfrac{1}{2}\int_a^{a''} f'(\tau)\,d\tau\frac{\partial}{\partial t}\log[(\mathfrak{s}_0\mathfrak{r})-c(t-\tau)].$$

It is easy to see from these expressions that $\mathfrak{a}$ and ϕ are infinite at time t only on that portion of O's first singular curve which is made up of electricity fired out between times a and a', and on that portion of O_1's second singular curve which is made up of electricity which reaches O between times a and a''. There is thus a cancelling of singularities on these portions of the singular curves which overlap. If the two sources have been active for the whole of time there is, however, an uncompensated electric charge at infinity on one of the singular curves, and an equal and opposite charge is associated permanently with the portions of the singular curves that join O and O_1.

When the two sources O and O_1 are consecutive, the above field reduces to the electromagnetic field of a point charge of electricity, for we may write

$$[x-\xi(a')-c(a_1-a')\,l(a')]^2+[y-\eta(a')-c(a_1-a')\,m(a')]^2$$
$$+[z-\zeta(a')-c(a_1-a')\,n(a')]^2 = c^2(t-a_1)^2,$$

and if we treat $a'-a$ and a_1-a' as small, this equation gives

$$(a'-a)[(\mathfrak{vr})-c^2(t-a)]+c(a_1-a')[(\mathfrak{sr})-c(t-a)] = 0,$$

where $\mathfrak{v}$ denotes the vector whose components are $\xi'(a)$, $\eta'(a)$, $\zeta'(a)$, and $\mathfrak{r}$ the vector whose components are $x-\xi(a)$, $y-\eta(a)$, $z-\zeta(a)$. In a first approximation $\mathfrak{s}$ can be used to denote the value of this vector at time a'.

In a similar way we find that

$$(\alpha''-\alpha)[(\mathrm{v}\mathrm{r})-c^2(t-\alpha)]-c\,(\alpha''-\alpha_1)\,[(\bar{\mathrm{s}}_0\mathrm{r})-c\,(t-\alpha'')] = 0,$$

where $\bar{\mathrm{s}}_0$ denotes the value of the vector s_0 at time α''.

Now to a first approximation

$$\mathrm{a} = \frac{c}{2}f'(\alpha)\,\frac{\mathrm{s}\,(\alpha'-\alpha)}{(\mathrm{s}\mathrm{r})-c(t-\alpha')} - \frac{c}{2}\,f'(\alpha)\,\frac{\bar{\mathrm{s}}_0(\alpha''-\alpha)}{(\mathrm{s}_0\mathrm{r})-c(t-\alpha'')}$$

$$= -\frac{c^2}{2}\,f'(\alpha)\,\frac{\mathrm{s}(\alpha_1-\alpha')+\bar{\mathrm{s}}_0(\alpha''-\alpha_1)}{(\mathrm{v}\mathrm{r})-c^2\,(t-\alpha)},$$

and we may also write

$$c\mathrm{s}(\alpha_1-\alpha')+c\bar{\mathrm{s}}_0\,(\alpha''-\alpha_1) = (\alpha''-\alpha')\bar{\mathrm{v}},$$

where $\bar{\mathrm{v}}$ denotes O's velocity at some instant between α' and α'', for this equation simply tells us that a light-particle fired from O at time α' is reflected from O_1 at time α_1, and comes back to a new position of O at time α''. We may also write

$$f(\alpha'')-f(\alpha') = 2T \quad \text{or} \quad (\alpha''-\alpha')f'(\alpha) = 2T,$$

to a first approximation; hence, finally, when we replace the above value of v by its value at time α,

$$\mathrm{a} = -cT\,\frac{\mathrm{v}}{(\mathrm{v}\mathrm{r})-c^2\,(t-\alpha)}, \qquad \phi = -cT\,\frac{c}{(\mathrm{v}\mathrm{r})-c^2\,(t-\alpha)}.$$

These expressions are, however, no other than Liénard's potentials for the field of a point charge of magnitude $4\pi cT$. Hence the two superposed radiant fields are equivalent in a suitable region to the field of an electric pole, and the potential energy in the field of the electric pole may be supposed to arise from the kinetic energy of the superposed radiant fields coupled with the fact that the cancelling of electricity tends to produce a steady state. The equation

$$c\mathrm{s}\,(\alpha_1-\alpha')+c\bar{\mathrm{s}}_0(\alpha''-\alpha_1) = (\alpha''-\alpha')\bar{\mathrm{v}}$$

connects the two directions of projection of fragments of electricity with the velocity of O. In the limit when O and O_1 are consecutive, it takes the form

$$c\mathrm{s}+kc\mathrm{s}_0 = (1+k)\mathrm{v},$$

where s, s_0 and v are all measured at the same instant α, and k is some function of α. The geometrical interpretation of this equation is that if lines OS, OS_0, OV of lengths c, c, v respectively are drawn in the directions of the vectors s, s_0, v, their extremities lie on a straight line.

Directions of projection which are associated in this way have already been considered.* Murnaghan has shown,† moreover, that if one direction of projection varies so that the fragments of electricity always lie on a line of electric force of an electric pole O, then the associated direction of projection is always associated with another line of electric force in the same way.

The manner in which the direction of projection must vary in order that the fragments of electricity may lie on a moving line of electric force has already been found.‡ The unit vector $\mathfrak{s}$ must satisfy the differential equation

$$(c^2-v^2)\frac{d\mathfrak{s}}{da} = [c-(\mathfrak{s}\mathfrak{v})]\frac{d\mathfrak{v}}{da} - \left(\mathfrak{s}\frac{d\mathfrak{v}}{da}\right)(c\mathfrak{s}-\mathfrak{v}). \qquad (4)$$

We shall now endeavour to give a geometrical interpretation of this equation.

6. *A Geometrical Theorem relating to Lines of Electric Force.*

Let us regard x, y, z and ict as rectangular coordinates in a space of four dimensions. Then the path of a moving electric pole is represented by a curve in the space S_4, and the path of a light-particle is represented by an isotropic line in this space.

Let P be a point on this curve, PQ an isotropic line through P, and P', P'' two points on the curve which are consecutive to P. We may draw a sphere through $PQP'P''$, and this sphere will contain the isotropic line PQ.

Let $P'Q'$ be an isotropic line consecutive to PQ which also lies on the sphere: then if P, P', P'', P''', are consecutive points on our curve we may draw a sphere through $P'Q'P''P'''$ and determine an isotropic line through P'' which lies on this new sphere, and so on.

Proceeding in this way we may assign an isotropic line for each point of the curve, and this means that we have a rule for assigning the direction of projection of a light-particle for each position of the electric pole.

Now a sphere in S_4 corresponds to a hyperboloid of revolution in ordinary space and an isotropic line on the sphere to a generator of the

---

* *Messenger of Mathematics*, May 1915.

† *Johns Hopkins Circular*, July 1915; *American Jour. of Math.*, April 1917.

‡ The equation was first given by Leigh Page, *American Journal of Science*, Vol. 38 (1914), p. 169. It was obtained independently by the author, *Bull. Amer. Math. Soc.*, March 1915.

hyperboloid. The rule tells us then that consecutive directions of projection are generators of a hyperboloid of revolution which passes through three consecutive positions of the moving electric pole.

Let the axis of revolution for this hyperboloid be taken as the axis of z: then the coordinates of an arbitrary point on the hyperboloid may be represented by

$$x = a\frac{w-u}{1-uw}, \quad y = a\frac{1+uw}{1-uw}, \quad z = a\cot\theta\frac{u+w}{1-uw},$$

$$ct = a\operatorname{cosec}\theta\frac{u+w}{1-uw},$$

where $u = \text{const.}$, $w = \text{const.}$ are the equations of the generators, a and θ being constants for the hyperbola.

The direction cosines of the two generators through a point (u, w) are

$$l = \frac{w^2-1}{w^2+1}\sin\theta, \quad m = \frac{2w}{w^2+1}\sin\theta, \quad n = \cos\theta,$$

$$l_0 = \frac{1-u^2}{1+u^2}\sin\theta, \quad m_0 = \frac{2u}{1+u^2}\sin\theta, \quad n_0 = \cos\theta\ ;$$

for we have

$$\lambda^2 dx = a(w^2-1)du + a(1-u^2)dw, \quad \lambda^2 dy = 2aw\,du + 2au\,dw,$$

$$\lambda^2\tan\theta\,dz = a(w^2+1)du + a(1+u^2)dw,$$

$$\lambda^2 c\sin\theta\,dt = a(w^2+1)du + a(1+u^2)dw,$$

where $$\lambda = 1-uw.$$

The component velocities of a point moving on the hyperboloid in such a way that it corresponds to a curve on the sphere in S_4 are

$$x' = c\sin\theta\frac{w'(1-u^2)-u'(1-w^2)}{w'(1+u^2)+u'(1+w^2)}, \quad y' = c\sin\theta\frac{2w'u+2uw'}{w'(1+u^2)+u'(1+w^2)},$$

$$z' = c\cos\theta,$$

where u' and w' denote the derivatives of u and w with respect to t. The component accelerations are

$$x'' = \frac{2\lambda c\sin\theta}{\mu^2}[(1+uw)(u'w''-u''w')+2u'w'(u'w-uw')],$$

$$y'' = \frac{2\lambda c\sin\theta}{\mu^2}[(u-w)(u'w''-u''w')+2u'w'(u'+w')],$$

$$z'' = 0,$$

where $$\mu = w'(1+u^2)+u'(1+w^2).$$

We may also deduce from the above equations that

$$c-lx'-my'-nz' = \frac{2c\lambda^2 w' \sin^2\theta}{\mu},$$

$$lx''+my''+nz''$$

$$= \frac{2c\lambda \sin^2\theta}{\mu^2}\left[\lambda(u''w'-u'w'')+2u'w'(u'w-uw')+\frac{4u'w'^2(u+w)}{1+w^2}\right],$$

$$cl-x' = -\frac{2\lambda w'c}{\mu}\frac{1+uw}{1+w^2}\sin\theta,$$

$$cm-y' = \frac{2\lambda w'c}{\mu}\frac{w-u}{1+w^2}\sin\theta,$$

$$cn-z' = 0.$$

Therefore

$$x''(c-lx'-my'-nz')-(cl-x')(lx''+my''+nz'') = \frac{16\lambda^2c^2\sin^3\theta\, ww'}{(1+w^2)^2\mu^2},$$

$$y''(c-lx'-my'-nz')-(cm-y')(lx''+my''+nz'') = \frac{8\lambda^2c^2\sin^3\theta\, w'(1-w^2)}{(1+w^2)^2\mu^2},$$

$$z''(c-lx'-my'-nz')-(cn-z')(lx''+my''+nz'') = 0.$$

Now $$c^2-x'^2-y'^2-z'^2 = \frac{4\lambda^2\sin^2\theta}{\mu^2},$$

$$l' = \frac{4ww'\sin\theta}{(1+w^2)^2}, \qquad m' = \frac{2w'(1-w^2)\sin\theta}{(1+w^2)^2}, \qquad n' = 0,$$

hence if $\mathfrak{s}$ denotes the vector with components l, m, n, and $\mathfrak{v}$ the vector with components (x', y', z'), we have the equation

$$(c^2-v^2)\frac{d\mathfrak{s}}{dt} = [c-(\mathfrak{v}\mathfrak{s})]\mathfrak{v}'-(c\mathfrak{s}-\mathfrak{v})(\mathfrak{s}\mathfrak{v}'),$$

which agrees with (4) of § 5.

The theorem that has just been proved may also be interpreted geometrically as follows :—

Let $P_1P_2P_3$ be three consecutive positions of the moving electric pole, and let P_1Q_1, P_2Q_2 be consecutive directions of projection associated with the same line of electric force. Then it is possible for light-particles which start from P_1, P_2 respectively at times τ_1, τ_2, to reach Q_1 and Q_2 at times t_1, t_2 respectively, while a light-particle which starts from Q_1 at time t_1 arrives at the points Q_2 and P_3 at times t_2, τ_3 respectively.

The light-particles are supposed to travel along straight lines with the velocity of light, and the theorem tells us that the light-particle which arrives at P_3 encounters on its way the light-particles which start from P_1 and P_2.

The fact that P_1Q_1, P_2Q_2 are generators of a hyperboloid of revolution which passes through P_3 gives an interesting meaning to associated directions of projection, for two associated directions of projection through P_1 are generators of the same hyperboloid of revolution. We also see that the light-particles which start from the points P_1, P_2, P_3, give one line of force, while the light-particles which come to P_1, P_2, P_3 will, if they continue their paths unbroken, give an associated line of force.

The circumstance that light-particles appear to flow both to and from the moving electric pole is exactly what we should expect when we consider the field of the electric pole to be built up in the manner described in §§ 4 and 5, for it may be recalled that, in the field wherein concentrated charges are created and shot out in only one direction which varies with the time, electricity must be supposed to flow towards the primary singularity as soon as the creation of electricity has been initiated. This electricity is indeed generated by a secondary creation or breaking up of doublets that is initiated by the free electricity that has already been formed, and this inward flow of electricity tends to secure the permanency of the electric charge associated with the electric pole.

It should be remarked that according to the present theorem a line of electric force in the field of an electric pole always consists of the same particles of electricity, but these particles are arranged so that positive and negative charges cancel, and so there is no volume distribution of electricity. The electric charges travel along straight lines with the velocity of light, and the law of variation of these lines is given by our geometrical theorem.

An alternative proof of this theorem is given in the paper referred to on p. 99.

7. *The Interference of Singular Radiant Fields of a Second Type.*

Let us now consider a singular radiant field specified by the potentials

$$\mathrm{B}-i\mathrm{A} = -ic\int_{-\infty}^{a} d\tau \nabla \frac{(\mathfrak{s}_1\mathfrak{r})-cp_1(t-\tau)}{(\mathfrak{s}\mathfrak{r})-c(t-\tau)},$$

$$\Psi-i\Phi = \quad i\int_{-\infty}^{a} d\tau \frac{\partial}{\partial t}\frac{(\mathfrak{s}_1\mathfrak{r})-cp_1(t-\tau)}{(\mathfrak{s}\mathfrak{r})-c(t-\tau)},$$

where the notation is the same as in § 4. To obtain a cancelling of singularities we must subtract from this a second singular radiant field whose primary singularity O_1 moves along a second curve C_1. This curve must be connected with that described by O in such a way that in each position O_1 is hit by a light-particle fired out from some associated position of O in the direction specified by the unit vector s. The functions of type s, s_1 and p_1 for O_1 must also have the same values as the corresponding functions at the corresponding point O. The quantities $(\mathrm{sr})-c(t-\tau)$, $(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)$ then have the same values at O_1 as at the associated point O; for if r_1 denote the radius vector to (x, y, z) from the position of O_1 at time τ_1, r the radius vector from the associated position of O, we may write

$$\mathrm{r}_1 = \mathrm{r}-c(\tau_1-\tau)\,\mathrm{s},$$

and the result follows at once when we make use of the identities

$$s^2 = 1, \qquad (\mathrm{ss}_1) = p_1.$$

The potentials for the second field may consequently be written in the form

$$\mathrm{B}-i\mathrm{A} = -ic\int_{-\infty}^{\alpha'} d\tau\nabla\,\frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)}{(\mathrm{sr})-c(t-\tau)},$$

$$\Psi-i\Phi = \quad i\int_{-\infty}^{\alpha'} d\tau\frac{\partial}{\partial t}\,\frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)}{(\mathrm{sr})-c(t-\tau)}.$$

Subtracting the two fields we obtain a field specified by the potentials

$$\mathrm{B}-i\mathrm{A} = -ic\int_{\alpha'}^{\alpha} d\tau\nabla\,\frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)}{(\mathrm{sr})-c(t-\tau)},$$

$$\Psi-i\Phi = \quad i\int_{\alpha'}^{\alpha} d\tau\frac{\partial}{\partial t}\,\frac{(\mathrm{s}_1\mathrm{r})-cp_1(t-\tau)}{(\mathrm{sr})-c(t-\tau)}.$$

This field has singularities only on a curve of finite length joining the two moving points O and O_1. It should be noticed that it is much easier to bring about a cancelling of singularities in the present case because each primary singular point has only *one* singular curve attached to it, so that the function p_1 can vary in an arbitrary manner, and is not restricted like the function $f(\tau)$ which occurs in § 5.

In a field of the present type the light-particles which are fired out from O may be regarded as either electric or magnetic doublets or a combination of both. This explains why it is not necessary for particles to be fired out in two directions as in the case when the particles are charged; it is easy, however, by a suitable combination of fields of this type to

obtain a field in which doublets are fired out in any number of directions from a moving point O.

It is quite possible, though not likely, that ultra-violet light and X-rays consist of a combination of two types of radiation, namely the ordinary radiation from an electron whose motion is accelerated, and a singular radiant field of the present type. The presence of the moving doublets would be in accordance with some ideas of Prof. W. H. Bragg* that have now been almost abandoned. The dual nature of X-rays was suggested by O. Heaviside† soon after Bragg had proposed the doublet theory, but Heaviside considered the case in which the corpuscles carried along by the waves are charged particles.‡ The singular radiant field would in this case be of the type considered in §§ 4 and 5.

If the dual nature of light and X-rays is ever considered to be established, the present analysis may be of some interest as it provides a possible means of accounting mathematically for the absorption or deflection of the corpuscular radiation by means of the phenomenon of the cancelling of singularities.

8. *Some Speculations on the Structure of Æther and Matter.*

To avoid circumlocution, a field in which there is a continual separation of electricity at a point O, which is either at rest or in motion, will be called *a field of Heaviside's type*. As in § 4, the two oppositely charged particles of electricity which are produced at any instant are supposed to move away from the source O with the velocity of light, describing rectilinear paths.

We have seen that a field of Heaviside's type can be regarded as the result of superposing two fields in which the volume distributions of electricity cancel. The component fields will be called *logarithmic fields*, and will be regarded here as more fundamental than the field of an electric pole, because they give some indication of the electrical nature of the æther and the manner in which electric charges are maintained; moreover, as we have already seen, the field of a moving electric pole may be regarded as a limiting case of four superposed fields of this type.

We shall now adopt the hypothesis that matter is composed of the primary singularities of logarithmic fields, and that the æther is produced by the electricity which is fired out from these sources after it has been

* *Proc. Royal Institution*, Jan. 1911.

† *Electromagnetic Theory*, Vol. 3, p. 477.

‡ Some cases of fields of this type have also been considered by Levi Civita, *Comptes rendus*, t. 145 (1907), pp. 417–420; *Rend. Lincei*, Vol. 18 (1909), p. 83.

created there. The æther must also be supposed to contain the electricity which is created by the electricity travelling outwards from the sources. This moving electricity seems indeed either to induce a separation of electric charges in the sense of a true creation, or to break up doublets which are already present in the æther on account of the existence of other sources. We are unable to decide at present which view is correct, as the true nature of electricity is not understood. If an electric charge is simply a place where some quantity is a maximum or minimum, then new electric charges can come into existence in pairs just as maxima and minima can be created together, and in this connection it is interesting to note that new maxima and minima often come into existence where a maximum or minimum already exists.

This analogy is, however, rather far fetched, and it seems more reasonable to assume that electric charges are created from electric doublets which existed previously.

In a field built up from a very large number of superposed logarithmic fields there will generally be a volume density of electricity and a convection current at each point of space. This can be avoided at points not occupied by matter if there is a complete cancelling of singularities, but we shall find it useful to consider the general case when the cancelling is not complete.

If ρ is the volume density of electricity and $\rho\mathrm{u}$ the convection current, the vectors E and H in the general field may be supposed to satisfy the equations

$$c\,\mathrm{rot}\,\mathrm{H} = \frac{\partial \mathrm{E}}{\partial t} + \rho\mathrm{u}, \qquad \mathrm{div}\,\mathrm{E} = \rho,$$

$$c\,\mathrm{rot}\,\mathrm{E} = -\frac{\partial \mathrm{H}}{\partial t}, \qquad \mathrm{div}\,\mathrm{H} = 0,$$

which occur in the ordinary theory of electrons. At a point where there is no matter ρ and $\rho\mathrm{u}$ may be supposed to refer to the free electricity which exists in the æther, while at points where there is matter they may be supposed to be composite, and we shall write

$$\rho = \rho_0 + \rho_1, \qquad \rho\mathrm{u} = \rho_0\mathrm{u}_0 + \rho_1\mathrm{u}_1,$$

where ρ_0 and $\rho_0\mathrm{u}_0$ are associated with the electricity which is connected with matter.

The function ρ_1 does not indicate the total amount of electricity of one sign that is contained in the æther; but, if we consider only those logarithmic fields in which *positive* electricity is produced in a concentrated form, the electric storage S for all these fields seems to be capable of

furnishing some information on this point. This function S is given by the formula of § 4, viz.,

$$S = -\tfrac{1}{2}\Sigma\frac{f'(\alpha)}{N} = -\frac{1}{c}\Sigma\Omega,$$

and is a solution of the wave equation at points not occupied by matter. This function can be found as soon as the structure of the electric poles is known, for this structure gives us the function $f'(\alpha)$, and so S can be determined.*

There is, of course, a function S_1 associated with the positive electricity of concentrated form, and a function S_2 associated with the negative electricity of concentrated form. In order that there may be no volume distribution of electricity in the æther, it is only necessary that the function

$$S = S_1 - S_2 = -\frac{1}{c}\Omega,$$

should be *constant* throughout the æther: it is not necessary that it should be zero.

When there is a volume distribution of electricity in the æther which may be derived from the function Ω by means of the formulæ

$$\rho\mathbf{u} = c\boldsymbol{\nabla}\Omega, \qquad \rho = -\frac{1}{c}\frac{\partial\Omega}{\partial t},$$

the fundamental equations may be satisfied by putting

$$\mathbf{H} = \operatorname{rot}\mathbf{A}, \qquad \mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi, \qquad \Omega = \operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t},$$

where the vertex $\mathbf{A}$ and the scalar Φ both satisfy the wave-equation. It is possible, however, that when there is not a complete cancelling of singularities of a concentrated form, the distribution of electricity in the æther may not be derivable from a single function Ω, and more complicated laws limiting the distribution of ρ and $\rho\mathbf{u}$ should therefore be studied. A case of great interest arises when $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{u}$ are connected by the relation

$$\mathbf{H} = \frac{1}{e}[\mathbf{uE}].$$

---

* Conversely when S is known, $f'(\alpha)$ may be determined for each point occupied by matter, and the total amount of electricity set free may thus be ascertained. It may be, however, that electric storage should be regarded as the fundamental entity and electric charge a quantity arising from its variation. The sign of the charge would then be positive or negative according as the storage increases or decreases.

Fields of this type were considered by Sir Joseph Thomson in his theory of moving Faraday tubes, and will consequently be called *fields of Thomson's type*. It may be shown that the vectors $\mathbf{E}$ and $\mathbf{H}$ in such a field can always be expressed in the form

$$\mathbf{E} = Z[\nabla X, \nabla Y], \qquad \mathbf{H} = \frac{z}{c}\frac{\partial X}{\partial t}\nabla Y - \frac{z}{c}\frac{\partial Y}{\partial t}\nabla X,$$

where X, Y, and Z are certain functions of x, y, z, and t.

9. *Fields of Thomson's Type.*

The proposition which we wish to prove may be stated as follows :—

If the coefficients in an integral form*

$$E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - cH_x d(x, t) - cH_y d(y, t) - cH_z d(z, t)$$

are connected by the three relations embodied in the vector equation

$$c\mathbf{H} = [\mathbf{uE}],$$

where $$\rho\mathbf{u} = c\,\mathrm{rot}\,\mathbf{H} - \frac{\partial \mathbf{E}}{\partial t}, \qquad \rho = \mathrm{div}\,\mathbf{E},$$

then the integral form can be expressed in the form $Zd(X, Y)$.

To prove this proposition we shall make use of the following theorem which has been demonstrated by A. C. Dixon† and C. Méray.‡

If the coefficients in the integral form are connected by the relations embodied in the vector equations

$$c\,\mathrm{rot}\,\mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}, \qquad \mathrm{div}\,\mathbf{E} = 0, \qquad (\mathbf{EH}) = 0,$$

then the integral form can be expressed in the form $d(X, Y)$.

This theorem tells us that the given integral form can be expressed in the form $Zd(X, Y)$ if a function $Z \equiv e^{\theta}$ can be found such that

$$c\,\mathrm{rot}\,(\mathbf{H}e^{-\theta}) = \frac{\partial}{\partial t}(\mathbf{E}e^{-\theta}), \qquad \mathrm{div}\,(\mathbf{E}e^{-\theta}) = 0,$$

for the equation $(\mathbf{EH}) = 0$ is satisfied in virtue of the relation $c\mathbf{H} = [\mathbf{uE}]$.

* The notation is the same as that used by A. C. Dixon, *Phil. Trans.* (A), Vol. 195 (1899), p. 151.

† *Loc. cit.*

‡ *Ann. de l'école normale*, Vol. 16 (1899), p. 509. See also V. Volterra, *Atti della R. Acc. dei Lincei*, Vol. 3, fasc. 10 (1887), and *The Rice Institute Pamphlet*, Jan. 1917.

The differential equations for θ may be written in the form

$$c[\nabla\theta, \mathbf{H}]-\mathbf{E}\frac{\partial\theta}{\partial t}=\rho\mathbf{u}, \qquad (\mathbf{E}, \nabla\theta)=\rho,$$

and if we substitute for $\mathbf{H}$ in the first equation it is clear that the equations for θ are equivalent to only two equations, viz.,

$$\frac{\partial\theta}{\partial t}+(\mathbf{u}\,.\,\nabla\theta)=0, \qquad (\mathbf{E}\,.\,\nabla\theta)=\rho.$$

The condition for the compatibility of these equations is that the equation

$$(\mathbf{E}\,.\,\nabla)\left[\frac{\partial\theta}{\partial t}+(\mathbf{u}\,.\,\nabla\theta)\right]-\left[\frac{\partial}{\partial t}+(\mathbf{u}\,.\,\nabla)\right][(\mathbf{E}\,.\,\nabla\theta)-\rho]=0,$$

should be a linear combination of the preceding equations. Now the equation of continuity gives

$$\frac{\partial\rho}{\partial t}+(\mathbf{u}\,.\,\nabla)\,\rho+\rho\,\mathrm{div}\,\mathbf{u}=0,$$

so that the preceding equation reduces to

$$(\mathbf{E}\,.\,\nabla)(\mathbf{u}\,.\,\nabla\theta)-(\mathbf{u}\,.\,\nabla)(\mathbf{E}\,.\,\nabla\theta)-\left(\frac{\partial\mathbf{E}}{\partial t}\nabla\theta\right)-\rho\,\mathrm{div}\,\mathbf{u}=0.$$

But $$(E\,.\,\nabla)\,\mathbf{u}-(\mathbf{u}\,.\nabla)\,\mathbf{E}-\mathbf{E}\,\mathrm{div}\,u+\mathbf{u}\,\mathrm{div}\,\mathbf{E}\equiv\mathrm{rot}\,[\mathbf{uE}],$$

and $$\mathrm{rot}\,[\mathbf{uE}]=c\,\mathrm{rot}\,\mathbf{H}=\frac{\partial\mathbf{E}}{\partial t}+\rho\mathbf{u}.$$

Hence the equation reduces to

$$(\mathrm{div}\,\mathbf{u})\,[(\mathbf{E}\,.\,\nabla\theta)-\rho]=0.$$

The condition for compatibility is thus satisfied.

It should be noticed that, when $\mathbf{E}$ and $\mathbf{H}$ are connected by the equation

$$c\rho\mathbf{H}=[\rho\mathbf{u}\,.\,\mathbf{E}],$$

it is necessary to add to this the equation

$$(\mathbf{EH})=0,$$

if we want the conditions that the integral form can be expressed in the form $Zd(X, Y)$. The conditions can, of course, be found from the theory of the invariants of a bilinear form.*

* This theory is developed by Christoffel, *Crelle's Journal*, Vol. 70 (1869), p. 46.

Let us now consider some of the properties of a field of Thomson's type. Since $(\mathbf{EH})=0$ and $c\,\mathrm{rot}\,\mathbf{E}=-\dfrac{\partial \mathbf{H}}{\partial t}$, $\mathrm{div}\,\mathbf{H}=0$, the integral form

$$H_x d(y,z)+H_y d(z,x)+H_z d(x,y)+cE_x d(x,t)+cE_y d(y,t)+cE_z d(z,t)$$

can be expressed in the form $d(V,\tau)$. Hence we may write

$$\mathbf{H}=[\nabla V,\ \nabla\tau],\qquad c\mathbf{E}=\frac{\partial\tau}{\partial t}\nabla V-\frac{\partial V}{\partial t}\nabla\tau. \tag{1}$$

But we may also write

$$\mathbf{H}=\frac{Z}{c}\frac{\partial X}{\partial t}\nabla Y-\frac{Z}{c}\frac{\partial Y}{\partial t}\nabla X,\qquad \mathbf{E}=Z[\nabla X,\nabla Y]: \tag{2}$$

hence it follows that

$$\left.\begin{aligned}(\nabla V,\nabla X)&=\frac{1}{c^2}\frac{\partial V}{\partial t}\frac{\partial X}{\partial t},\qquad (\nabla V,\nabla Y)=\frac{1}{c^2}\frac{\partial V}{\partial t}\frac{\partial Y}{\partial t},\\ (\nabla\tau,\nabla X)&=\frac{1}{c^2}\frac{\partial\tau}{\partial t}\frac{\partial X}{\partial t},\qquad (\nabla\tau,\nabla Y)=\frac{1}{c^2}\frac{\partial\tau}{\partial t}\frac{\partial Y}{\partial t}.\end{aligned}\right\} \tag{3}$$

If we regard x, y, z and ict as the rectangular coordinates of a point in a space of four dimensions, these equations imply that the surfaces $V=\text{const.}$, $\tau=\text{const.}$ are orthogonal to the surfaces $X=\text{const.}$, $Y=\text{const.}$

It should be noticed that the equations $X=\text{const.}$, $Y=\text{const.}$ represent a moving line of electric force, while the equations $V=\text{const.}$, $\tau=\text{const.}$ represent a moving line of magnetic force. Hence in a field of Thomson's type, the moving lines of electric and magnetic force have, as their images in a space of four dimensions, two systems of surfaces such that any member of one family is orthogonal to any member of the other.

One way of obtaining two families of surfaces of this type is to take a quadruply orthogonal system of hyper-surfaces (varieties) in S_4. If the equations of the hyper-surfaces are $X=\text{const.}$, $Y=\text{const.}$, $V=\text{const.}$, $\tau=\text{const.}$, respectively, then the surfaces $X=\text{const.}$, $Y=\text{const.}$ are orthogonal to the surfaces $V=\text{const.}$, $\tau=\text{const.}$

It is known that any singly infinite system of hyperspheres belongs to a quadruply orthogonal system,* and that the hyperspheres are mapped conformally on one another by their orthogonal trajectories. The remain-

* See Darboux, *Leçons sur les systèmes orthogonaux.*

ing members of the quadruply orthogonal system can be determined by finding a triply orthogonal system of surfaces in one hypersphere; the hyper-surface formed by drawing orthogonal trajectories through points of one of these surfaces then belongs to the quadruply orthogonal system.

Now it is known that any singly infinite system of spheres on a hypersphere forms part of a triply orthogonal system, and the orthogonal trajectories through the points of one of these spheres will cut all the other hyperspheres in spheres. Hence it is possible to find two orthogonal systems of ∞^2 surfaces in which one system consists entirely of spheres.

Electromagnetic fields in which the moving lines of magnetic force are represented in S_4 by spheres are of special interest. The field of a moving point charge is of this type, for R. Hargreaves has obtained a representation of the field in the form (1), where the functions V and τ are defined as follows :—

$$[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2 = c^2(t-\tau)^2 \quad (\tau < t), \qquad (4)$$

$$V = e\,\frac{\xi''(\tau)[x-\xi]+\eta''(\tau)[y-\eta]+\zeta''(\tau)[z-\zeta]+c^2-\xi'^2(\tau)-\eta'^2(\tau)-\zeta'^2(\tau)}{\xi'(\tau)[x-\xi]+\eta'(\tau)[y-\eta]+\zeta'(\tau)[z-\zeta]-c^2[t-\tau]}$$

$$\equiv e\,\frac{K}{M},$$

say.

The functions X and Y may be obtained by solving the equations (3). If $F(w, \tau) = \text{const.}$ is a solution of the Riccatian equation

$$2\mu\,\frac{dw}{d\tau} = [\xi'+i\eta'-w(\xi'+c)][w(\xi''-i\eta'')+\xi'']$$
$$-[w(\xi'-i\eta')+\xi'-c][\xi''+i\eta''-w\xi''], \qquad (5)$$

where $$\mu = c^2-\xi'^2-\eta'^2-\xi'^2,$$

we may write $$X+iY = F\left[\frac{x-\xi(\tau)+i\{y-\eta(\tau)\}}{z-\zeta(\tau)+c(t-\tau)},\ \tau\right].$$

Let us put $X+iY = \sigma$. Then a more general field of Thomson's type may be obtained by writing

$$\mathbf{H} = [\nabla W,\ \nabla\tau], \qquad c\mathbf{E} = \frac{\partial\tau}{\partial t}\nabla W - \frac{\partial W}{\partial t}\nabla\tau,$$

where $$W = V+Rf(\sigma, \tau).$$

This field may be regarded as the sum of the field of a point charge and a singular radiant field. To obtain the lines of electric force, we must express

$$Zd(X, Y)+Id(f, \tau)$$

in the form $Z_0 d(X_0, Y_0)$. Now since Maxwell's equations are satisfied, Z is simply a function of X and Y, and by a proper choice of F it can be made equal to unity. Let us write $\bar{\sigma}$ for $X-iY$; then the above expression is

$$\frac{1}{2i}\left[f'(\sigma, \tau)\, d(\sigma, \tau) - f'(\bar{\sigma}, \tau)\, d(\bar{\sigma}, \tau) - d(\sigma, \bar{\sigma})\right],$$

or

$$\frac{1}{2i} d(X_0, Y_0),$$

where $X_0 = \text{const.}$ and $Y_0 = \text{const.}$ are solutions of the equations

$$f'(\sigma, \tau)\, d\tau - d\bar{\sigma} = 0, \qquad f'(\sigma, \tau)\, d\tau - d\sigma = 0.$$

The lines of electric force in the present case are thus determined by the equations

$$\frac{d\sigma}{d\tau} = f'(\bar{\sigma}, \tau), \qquad \frac{d\sigma}{d\tau} = f'(\sigma, \tau),$$

which may be compared with the differential equations of the paths of particles of fluid in an irrotational motion of an incompressible fluid in two dimensions, X and Y being the coordinates of a particle at time τ.

10. *A Simple Generalisation of the Field of a Point Charge.*

It has been found by trial that if Hargreaves' function V is generalised by writing

$$V = \frac{\alpha(\tau)[x-\xi]+\beta(\tau)[y-\eta]+\gamma(\tau)[z-\zeta]+\mu(\tau)}{\xi'(\tau)[x-\xi]+\eta'(\tau)[y-\eta]+\zeta'(\tau)[z-\zeta]-c^2(t-\tau)} \equiv \frac{L}{M},$$

say, where $\alpha(\tau)$, $\beta(\tau)$, $\gamma(\tau)$, and $\mu(\tau)$ are arbitrary functions of τ, and τ is defined as before, it is still possible to determine two functions X and Y so that the equations (3) are satisfied, but Z is no longer a function of X and Y only. The Riccatian equation which replaces (5) is

$$2\mu \frac{dw}{d\tau} = [\xi'+i\eta'-w(\zeta'+c)][w(\alpha-i\beta)+\gamma] - [w(\xi'-i\eta')+\zeta'-c](\alpha+i\beta-w\gamma). \quad (6)$$

This may be regarded as the general Riccatian equation with complex coefficients which are arbitrary functions of the independent variable τ.

Just as in the case of the field of a point charge, the lines of electric force in the field of Thomson's type, specified by the functions V and τ, can be regarded as the loci of light-particles fired out in directions specified by vectors of type s, which vary with τ according to a simple law. If a

and $\mathfrak{v}$ denote the vectors with components (α, β, γ), (ξ', η', ζ'), this law is in the present case

$$\mu\mathfrak{s}' = \mathfrak{a}\{c-(\mathfrak{s}\mathfrak{v})\}+\{\mathfrak{v}-c\mathfrak{s}\}(\mathfrak{a}\mathfrak{s}) \qquad (7)$$

The solution of this vector equation can of course be made to depend on the solution of the above Riccatian equation. If (l, m, n) are the components of $\mathfrak{s}$, the appropriate substitution is

$$w = \frac{l+im}{1+n}.$$

Let us now consider some further properties of the field. The component of the electric intensity along the radius from (ξ, η, ζ, τ) to an associated point (x, y, z, t) is easily found to be

$$E_r = \frac{1}{c^2}\frac{\partial V}{\partial t}\frac{\partial \tau}{\partial t} - (\nabla V, \nabla\tau) = \frac{\mu}{M^2}.$$

Comparing this with the corresponding expression for a point charge of Liénard's type, we see that the electric charge associated with the point singularity (ξ, η, ζ, τ) remains constant if μ is a constant multiple of c^2-v^2, but not otherwise.

It appears that electricity is generally fired out from the singularity (ξ, η, ζ, τ), for we have

$$\rho\mathfrak{u} = c\Gamma\nabla\tau, \qquad \rho = -\frac{1}{c}\Gamma\frac{\partial\tau}{\partial t},$$

where

$$\Gamma = \frac{2(\mathfrak{a}\mathfrak{v})+\mu'}{M^2}+\frac{2(c^2-v^2)L-2\mu K}{M^3}.$$

Now

$$M\frac{\partial\tau}{\partial x} = x-\xi, \qquad M\frac{\partial\tau}{\partial y} = y-\eta, \qquad M\frac{\partial\tau}{\partial z} = z-\zeta, \qquad M\frac{\partial\tau}{\partial t} = -c^2(t-\tau).$$

Hence at points (x, y, z, t) associated with (ξ, η, ζ, τ) the electricity moves away from the latter point with the velocity of light.

If $v^2 < c^2$, $\partial\tau/\partial t$ is positive, and so the sign of ρ is determined by the sign of Γ. Now Γ may be written in the form

$$M^3\Gamma = p(x-\xi)+q(y-\eta)+r(z-\zeta)-cs(t-\tau),$$

and, if μ is a constant multiple of c^2-v^2, we have

$$p\xi'+q\eta'+r\zeta' = cs.$$

This last equation implies that $p^2+q^2+r^2 > s^2$, for if (ξ', η', ζ') are regarded as the coordinates of a point within the sphere $x^2+y^2+z^2 = c^2$,

the condition implies that the point $(cp/s, cq/s, cr/s)$ lies on the polar plane of (ξ', η', ζ'), and so lies outside the sphere.

Since $p^2+q^2+r^2 > s^2$, Γ is zero over one half of a right circular cone generated by lines making a constant angle with the line whose direction cosines are proportional to (p, q, r). The vertex of this cone is at the point ξ, η, ζ, τ. We thus have the following result.

At each instant the moving singularity fires out electric charges of one sign in directions lying within the associated half-cone, and electric charges of the opposite sign in directions lying outside the half-cone. Equal amounts of positive and negative charges are shot out at each instant, but this statement needs modification if μ is not a constant multiple of c^2-v^2.

It should be noticed incidentally that a line of electric force always consists of the same particles of electricity. This is in accordance with a general theorem that has been proved for fields of Thomson's type.*

Let us next determine the rate at which energy is radiated to infinity from the moving singularity.

The component of Poynting's vector along the radius from (ξ, η, ζ, τ) to (x, y, z, t), is found to be

$$c\left[(\nabla V)^2 (\nabla\tau)^2-(\nabla V . \nabla\tau)^2\right].$$

Dropping all the terms which vanish at infinity to an order higher than $1/r^2$, we find that to this order of approximation the component is

$$\frac{c^2(t-\tau)^2}{M^6}\left[a^2M^2-2MN\,(\mathrm{a}\nu)+N^2(v^2-c^2)\right],$$

where $$N = \alpha\,(x-\xi)+\beta\,(y-\eta)+\gamma\,(z-\zeta).$$

Comparing this with the corresponding expression for the component of Poynting's vector in the field of a moving point charge of Liénard's type, we see that we need only write $\alpha, \beta, \gamma, \mu$ in place of $e\xi'', e\eta'', e\zeta'', e(c^2-v^2)$ respectively, in the formula for the rate of radiation of energy from the point charge, in order to obtain the formula for the rate of radiation in the present case.

It should be noticed that, if α, β and γ are zero, there is no loss of energy by radiation. It also follows from (7) that in this case the direction of projection of light-particles which form a line of electric force does not vary with the time, and so the corresponding portion of the line of electric force is straight.

Thus in a field of the present type, radiation of energy from the

* *Messenger of Mathematics*, Vol. 46, p. 136.

singularity ξ, η, ζ, τ appears to be associated with a change in form of the lines of electric force issuing from the singularity. In other words, the lines of force must vibrate to produce light.

It is important to notice that α, β and γ may be zero, although the acceleration of the singular point is not, so that in a field of the present type it is possible for the singular point to make revolutions round a closed curve without losing energy by radiation. This result may be of some interest in connection with Bohr's theory of atomic structure.

11. *Infinitesimal Transformations associated with Lines of Force.*

In analytical dynamics the idea of representing the motion of a system of particles by a succession of infinitesimal transformations is a familiar one. A similar idea can be used with advantage in the theory of moving Faraday tubes. We must endeavour to find an infinitesimal transformation which will send the singularity (ξ, η, ζ, τ) into a consecutive position, and each direction of projection of a light-particle into a corresponding direction of projection of a light-particle, the two directions being associated with the same line of electric force. We shall, moreover, choose the transformation so that a light-particle travelling along a straight line with the velocity of light corresponds to another light-particle travelling along a straight line with the velocity of light. The infinitesimal transformation and the motion of the light-particles may then be regarded as two permutable motions, both of which are possible motions for particles which always belong to the same Faraday tube. It is known that a spherical wave transformation establishes a correspondence between points which describe straight lines with the velocity of light. Now if r denotes the radius vector from the origin of coordinates to a point with coordinates (x, y, z), and $d\epsilon$ is a small quantity whose square may be neglected, the transformation

$$\left.\begin{aligned} \mathrm{r}_0 &= \mathrm{r} + d\epsilon[(r^2 - c^2t^2)\,\mathrm{P} + 2\lambda\mathrm{r} + 2[\mathrm{gr}] - 2\mathrm{p}ct + \mathrm{Q}], \\ ct_0 &= ct + d\epsilon[(r^2 - c^2t^2)S + 2\lambda ct - 2(\mathrm{pr}) + T], \\ \lambda &\equiv k - (\mathrm{Pr}) + Sct, \end{aligned}\right\} \qquad (1)$$

is an infinitesimal spherical wave transformation if the vectors P, Q, g, p, and the scalars k, S, T are constants.

Instead of regarding them as constants, however, we shall regard them as functions of a quantity ϵ, which is itself a function of the quantity τ, which is defined in terms of x, y, z and t in the same way as before. We

then have a succession of infinitesimal transformations, but the transformation associated with a particular value of τ is supposed to be applied only to those values of x, y, z and t that are associated with this value of τ.

Giving arbitrary increments δx, δy, δz, δt to x, y, z, t, which are consistent with the invariance of τ, the corresponding increments δx_0, δy_0, δz_0, δt_0 are determined by

$$\delta \mathrm{r}_0 = \delta \mathrm{r}(1+2\lambda d\epsilon)+2d\epsilon\{\mathrm{P}(\mathrm{r}\delta\mathrm{r})-\mathrm{P}c^2t\delta t-\mathrm{r}(\mathrm{P}\delta\mathrm{r})+[\mathrm{g}\delta\mathrm{r}]+cS\mathrm{r}\delta t-\mathrm{p}c\delta t\},$$

$$c\delta t_0 = c\delta t(1+2\lambda d\epsilon)+2d\epsilon\{S(\mathrm{r}\delta\mathrm{r})-ct(\mathrm{P}\delta\mathrm{r})-(\mathrm{p}\delta\mathrm{r})\}.$$

It is easy to verify by means of these equations that $(\delta r_0)^2 = c^2(\delta t_0)^2$ is a consequence of $(\delta r)^2 = c^2(\delta t)^2$, so that a point moving with the velocity of light corresponds to a point moving with the velocity of light. Let us now write $\delta\mathrm{r} = \mathrm{s}c\delta t$, $\delta\mathrm{r}_0 = \mathrm{s}_0 c\delta t_0$, where s and s_0 are unit vectors; then*

$$\mathrm{s}_0 = \frac{\mathrm{s}+2d\epsilon[\lambda\mathrm{s}+\mathrm{P}(\mathrm{sr})-\mathrm{r}(\mathrm{Ps})+[\mathrm{gs}]-\mathrm{P}ct+S\mathrm{r}-\mathrm{p}]}{1-2d\epsilon[(\mathrm{sP})ct-S(\mathrm{sr})+(\mathrm{sp})-\lambda]}.$$

Putting $\mathrm{s}_0 = \mathrm{s}+\frac{d\mathrm{s}}{dt}dt+\ldots$, we find on neglecting squares and higher powers of $d\epsilon$ that

$$\frac{dt}{d\epsilon}\frac{d\mathrm{s}}{dt} = 2\mathrm{s}\{(\mathrm{sP})ct-S(\mathrm{sr})+(\mathrm{sp})\}+2\mathrm{P}(\mathrm{sr})-2\mathrm{r}(\mathrm{sP})+2[\mathrm{gs}] \\ -2\mathrm{P}ct+2S\mathrm{r}-2\mathrm{p}.$$

Now put $x=\xi$, $y=\eta$, $z=\zeta$, $t=\tau$, and compare the resulting expression for $d\mathrm{s}/d\tau$ with that given by equation (7) of § 10. We see that the two expressions are identical if

$$2\mu\epsilon'(S\bar{\mathrm{r}}-\mathrm{P}c\tau-\mathrm{p}) = c\mathrm{a}, \qquad 2\mu\epsilon'\{[\mathrm{P}\bar{\mathrm{r}}]-\mathrm{g}\} = [\mathrm{va}], \tag{2}$$

where ϵ' is written for $d\epsilon/d\tau$, and $\bar{\mathrm{r}}$ is the radius vector from the origin to the point (ξ, η, ζ).

In order that the point ξ, η, ζ, τ may be transformed into $\xi+d\xi$, $\eta+d\eta$, $\xi+d\xi$, $\tau+d\tau$ by the infinitesimal transformation, we must have

$$\left.\begin{aligned} \mathrm{v} &= \epsilon'[(\bar{r}^2-c^2\tau^2)\mathrm{P}+2\bar{\lambda}\bar{\mathrm{r}}+2[\mathrm{g}\bar{\mathrm{r}}]-2\mathrm{p}c\tau+\mathrm{Q}], \\ c &= \epsilon'[(\bar{r}^2-c^2\tau^2)S+2\bar{\lambda}c\tau-2(\mathrm{p}\bar{\mathrm{r}})+T], \quad \bar{\lambda} = k-(\mathrm{P}\bar{\mathrm{r}})+Sc\tau. \end{aligned}\right\} \tag{3}$$

* It is easy to verify that the value of s_0 is unaltered if we replace r and t by $\mathrm{r}+c\mathrm{s}\omega$, and $t+\omega$ respectively, where ω is arbitrary. This means that a point which moves along a straight line with the velocity of light corresponds to a point moving along a straight line with the velocity of light.

If $\mathrm{P}\epsilon'$, $S\epsilon'$ and $k\epsilon'$ are chosen arbitrarily, the above equations will determine $\epsilon'\mathrm{p}$, $\epsilon'\mathrm{g}$, $\epsilon'\mathrm{Q}$, and $\epsilon' T$, so that the motion of the lines of electric force in the electromagnetic field may be described by means of the succession of infinitesimal transformations.

Let us next regard the infinitesimal transformations as given and endeavour to find the type of electromagnetic field to which they are applicable, with the aim of ascertaining when the field is the field of a point charge of Liénard's type.

In order that it may be possible to solve equations (2) for a, the relations

$$(\mathrm{v}[\mathrm{P}\bar{\mathrm{r}}]) = (\mathrm{vg}), \qquad c[\mathrm{P}\bar{\mathrm{r}}] - c\mathrm{g} + [\mathrm{vp}] - S[\mathrm{v}\bar{\mathrm{r}}] = 0, \tag{4}$$

must be satisfied. Substituting in these equations the values of v and c given by (3), we obtain four equations for ξ, η, ζ, τ, which may or may not be satisfied identically. The case in which the equations are satisfied identically is of chief interest, and it is easy to verify that the requirements are satisfied when the coefficients in the transformation are connected by the relations

$$2k\mathrm{p} = T\mathrm{P} - S\mathrm{Q}, \qquad 2k\mathrm{g} = [\mathrm{QP}]. \tag{4}$$

We shall call a transformation of this type a *normal transformation*.

12. *Coordinated Motions of Point Charges.*

Let us consider the properties of a system of electromagnetic fields of the type considered in § 10 when the singular points move along the path curves of a succession of infinitesimal normal transformations.

Let E and E' be the positions of two of the singular points for a given value of ϵ. Then if particles, fired from these positions at the appropriate times, collide after travelling along straight lines with the velocity of light, this property will be invariant under the infinitesimal transformation which sends E and E' into their consecutive positions. Since, moreover, the consecutive direction of projection derived by the infinitesimal transformation corresponds to the same line of electric force as the original direction of projection, the preceding result indicates that there is permanent incidence between certain lines of electric force issuing from E and E'.

When E and E' are given, the possible points of collision lie on a quadric of revolution with E and E' as foci; hence there is permanent incidence between each line of force from E and at least one line of force

from E'. When this relation holds, the two electromagnetic fields with E and E' as singularities will be said to be *partially incident*. For *complete incidence* there must be permanent incidence between each line of force from E and an infinity of lines of force from E'. This case arises when the infinitesimal transformation is the same for all values of ϵ, because then we can apply the above reasoning to *any* two positions of E and E'.

Let us now consider the hypothesis that the different point charges composing an electron move in such a way that there is partial incidence between the fields of each pair of point charges. This condition may be satisfied by making the point charges move along the path curves of a succession of normal infinitesimal transformations, provided the vector a determined by the transformations is proportional to the acceleration of the point charge at the point under consideration. An appropriate succession of infinitesimal transformations may be obtained as follows.

Let (x, y, z, ict) be regarded as the rectangular coordinates of a point in a space of four dimensions S_4, and let the path curves of our succession of infinitesimal transformations be represented in S_4 by the orthogonal trajectories of a family of hyperspheres (or hyperplanes)

$$b(\epsilon)[r^2-c^2t^2]-2(\mathrm{rf})+2h(\epsilon)c^2t+\theta(\epsilon)=0,$$

where the vector f is a function of ϵ. If derivatives with respect to ϵ are denoted by dashes, the orthogonal trajectories are given by the equations

$$\frac{d\mathrm{r}}{b\mathrm{r}-\mathrm{f}}=\frac{dt}{bt-h}=\frac{d\epsilon[b'(r^2-c^2t^2)-2(\mathrm{rf}')+2h'c^2t+\theta']}{2(b\theta+c^2h^2-f^2)},$$

and the succession of infinitesimal transformations by

$$\mathrm{r}_0=\mathrm{r}+\frac{d\epsilon}{2\phi}[(b\mathrm{r}-\mathrm{f})\{b'(r^2-c^2t^2)-2(\mathrm{rf}')+2h'c^2t+\theta'\}$$
$$-(b'\mathrm{r}-\mathrm{f}')\{b(r^2-c^2t^2)-2(\mathrm{rf})+2hc^2t+\theta\}],$$

$$t_0=t+\frac{d\epsilon}{2\phi}[(bt-h)\{b'(r^2-c^2t^2)-2(\mathrm{rf}')+2h'c^2t+\theta'\}$$
$$-(b't-h')\{b(r^2-c^2t^2)-2(\mathrm{rf})+2hc^2t+\theta\}],$$

where $$\phi=b\theta+c^2h^2-f^2.$$

Comparing these equations with (1), we see that the transformation is of the required type, and that

$$\left.\begin{array}{lll} 2\phi\mathrm{P}=b\mathrm{f}'-b'\mathrm{f}, & 2\phi\mathrm{g}=[\mathrm{f}'\mathrm{f}], & 2\phi\mathrm{p}=c(h'\mathrm{f}-h\mathrm{f}'), \\ 2\phi\mathrm{Q}=\theta\mathrm{f}'-\theta'\mathrm{f}, & 2\phi S=c(bh'-b'h), & 4\phi k=b\theta'-b'\theta, \\ & 2\phi T=\theta h'-\theta' h. & \end{array}\right\} \quad (6)$$

It is easy to verify, moreover, that the conditions for normality are satisfied.

Now we are interested in the motions of points (ξ, η, ζ, τ) which for each value of ϵ satisfy the equation

$$[\bar{r}^2 - c^2\tau^2] - 2(\bar{\mathrm{r}}\mathrm{f}) + 2hc^2\tau + \theta = 0.$$

For these points we have

$$\mathrm{v} = \frac{b\bar{\mathrm{r}} - \mathrm{f}}{b\tau - h}, \qquad \mu = e(c^2 - v^2) = \frac{e\phi}{(b\tau - h)^2}.$$

Equations (2) and (6) now give

$$\begin{aligned} \mathrm{a} &= \frac{e\epsilon'}{(b\tau - h)^2}\{\bar{\mathrm{r}}(bh' - b'h) - \tau(b\mathrm{f}' - b'\mathrm{f}) - (h'\mathrm{f} - h\mathrm{f}')\} \\ &= e\frac{d}{d\tau}\left\{\frac{b\bar{\mathrm{r}} - \mathrm{f}}{b\tau - h}\right\} - \frac{eb\mathrm{v}}{b\tau - h} + \frac{be(b\bar{\mathrm{r}} - \mathrm{f})}{(b\tau - h)^2} \\ &= e\mathrm{v}', \end{aligned}$$

where v' denotes the acceleration of the moving point (ξ, η, ζ, τ).

Since $\mathrm{a} = e\mathrm{v}'$ and $\mu = e(c^2 - v^2)$, the electromagnetic field with (ξ, η, ζ, τ) as moving singularity is the field of a point charge.

The type of coordinated motion at present under consideration has already been discussed in connection with a generalisation of Lorentz's electron,* and it has been shown that, if for any given value of ϵ the surface of the electron is represented in S_4 by a sphere, it will be represented by a sphere for all values of ϵ.

13. *Some Thoughts on Gravitation and the Apparent Constancy of the Electronic Charge.*

Let us consider two fields of Heaviside's type whose singular curves overlap without producing a complete cancelling of singularities. If the two primary singularities O and O_1 are very close together, we shall have a variable electric charge associated with the short interval OO_1, and at any instant this is equal but opposite in sign to the total charge associated at that instant with the two singular curves. It is possible also to imagine a case in which each singular curve carries the same charge, and indeed such an arrangement seems a very natural one on account of symmetry.

* *American Journal*, Vol. 34 (1912), p. 325.

We may imagine then a type of electron whose charge fluctuates round a mean value.* Two singular curves must be supposed to be attached to the electron, and we shall suppose that when the charge on the electron is $2f(t)$ the charge on each singular curve is $-f(t)$.

If Q is any point on a singular curve and τ is the instant at which a fragment of electricity must leave O in order to arrive at Q at time t, the charge associated at time t with the portion from Q to infinity of the singular curve is $-f(\tau)$. Let us now consider a second electron or positive nucleus of an atom at Q, and suppose that one of its singular curves coincides with the portion from Q to infinity of O's singular curve. Then there will be a complete cancelling of singularities on this portion of the singular curve if Q's singular curve carries a charge $f(\tau)$ at time t. If Q, like O, has two singular curves attached to it and the charges carried by them at any instant are equal, then the charge associated with Q at time t must be $-2f(\tau)$, and, like the charge associated with O, it must also fluctuate round a mean value. If now the extent of the fluctuation is negligibly small, the mean value of the charges at O and Q will be approximately equal, but of opposite signs.

The second singular curve from Q may be supposed to go to some other electron P, whose charge at time t_1 will be likewise $2f(\tau)$, t_1 being the time associated with t. Similarly O's second singular curve may be supposed to be a continuation of one from another point N, whose charge at time t_0 is $-2f(\tau)$, and so on.

It is not necessary, of course, to limit the number of singular curves issuing from an electron to two, for we may imagine the field of an electron or positive nucleus to be built up from any number of fields of Heaviside's type in which the primary singularities are very close together. We may, in fact, imagine a type of electron with singular curves extending in all directions, and may assume that the electric charge associated with each singular curve at any instant is a constant fraction of the charge associated with the electron at that instant. If now all the electrons and positive nuclei in the universe are connected together by singular curves so that there is a cancelling of singularities which will prevent electricity from going to infinity, it is easy to see from the foregoing analysis that there may be a mathematical reason why all the electrons in the universe carry the same mean charge.

It should be mentioned that it is not necessary to suppose that all the particles in the universe are connected together by singular curves made up of uncompensated electric charges, for we may imagine that when a

* The question whether an electron can lose some of its charge was raised by Heaviside (*l.c.*).

neutral "æther particle" is produced by a "cancelling of singularities" or a union of oppositely charged fragments of electricity, this æther particle breaks up again into the same oppositely charged fragments at a source of a singular radiant field. Let us then consider three sources O, O_1, O_2 of singular radiant fields, and suppose that the first two fields interfere with one another and furnish "æther particles" which break up at the third source.

Let a fragment of electricity start from O at time τ, let it arrive at O_1 at time τ_1 and combine with an oppositely charged fragment of electricity: finally let it part from its companion at O_2 at time τ_2. It is clear then that if the æther particles which break up at O_2 have been furnished by O and O_1 since the beginning of time, the charge associated with one of O_2's singular curves is equal at time τ_2 to the charge associated with one of O's singular curves at time τ.

Now the charge associated with O's singular curves at any instant may be supposed to determine in some way the charge at this instant on the electron or positive nucleus of which O forms part. Similar remarks apply to O_1 and O_2, and so there may be a definite relation between the magnitudes of the charged particles containing O, O_1 and O_2 respectively, whether these particles carry charges of the same or opposite signs.

This idea of the fluctuation of the electric charge associated with a particle of matter leaves a road open for a possible explanation of gravitation. Let us consider two particles A and B at associated times τ and t, that is at times such that a light-particle starting from A at time τ would arrive at B at time t. We shall suppose that the charge on A at time τ differs from its mean or normal value e by $\kappa\psi(\tau)$, and that the charge on B at time t differs from its normal value e' by $-\kappa_1\psi(\tau)$ where κ and κ_1 are constants. This means that the variation in B's charge is proportional to the variation which would be produced if a percentage of the charge fired out from A went direct to B.

If we now suppose that $\psi(\tau)$ varies periodically with a period $2T$, the mean value of the electric force exerted by A on B will be, according to the inverse square law,

$$\frac{1}{2r^2T}\int_0^{2T}[e+\kappa\psi(\tau)][e'-\kappa_1\psi(\tau)]\,d\tau.$$

Now $\int_0^{2T}\psi(\tau)\,d\tau = 0$, if e is the mean value of A's charge, and so the value of the above integral is

$$\frac{1}{r^2}\left(ee'-\frac{\lambda}{2}\kappa\kappa_1\right),$$

where $$\lambda = \frac{1}{2T}\int_0^{2T} [\psi(\tau)]^2 \, d\tau,$$

and is positive. Now, if κ and κ_1 are both positive, as we are entitled to assume, there will be an attraction in addition to the electrostatic force, whether e and e' are of the same or opposite signs. It is easy to see that the variation of the electric charge on an electron or positive nucleus which is necessary to account for gravitation is exceedingly small, perhaps of the order of $\pm 10^{-10}$ of the electronic charge during fractions 10^{-22} of the line $2T$, but this is simply a guess and may be wide of the mark.

The idea that gravitation may be due to a periodic disturbance is not new. Hicks,* Whittaker,† Korn,‡ and others, have suggested theories of this type. The present suggestion is also analogous to the well-known theory in which the attraction between two charges of opposite sign is supposed to be greater than the repulsion between two similar charges of the same magnitudes as the others. This theory, which is generally attributed to Mosotti, is mentioned in Maxwell's *Electricity and Magnetism*. It has been discussed recently by Lorentz,§ Gans,‖ J. J. Thomson,¶ and D. L. Webster.**

It should be noticed that if two charges move in such a way that at associated times they are at a constant distance apart, and if they are doubly connected by overlapping singular curves along which there is a cancelling of singularities, so that the charges on their singular curves are connected by relations of type

$$f(\tau) = \phi(t), \qquad c(t-\tau) = a = cT,$$

say, and $$f(\tau') = \phi(t), \qquad c(\tau'-t) = a = cT,$$

then $$f(\tau+2T) = f(\tau),$$

and so $f(\tau)$ is a periodic function with period $2T$. Similarly $\phi(\tau)$ is a periodic function with period $2T$. We have assumed here that at any instant the two singular curves from each charge carry the same amount of electricity. If the electricity which has been fired away from one of

* *Proc. Camb. Phil. Soc.*, Vol. 3 (1879), p. 276; Vol. 4 (1880), p. 29.

† *Monthly Notices Royal Astr. Soc.*, Vol. 62 (1902), p. 617.

‡ *Eine Theorie der Gravitation und der elect. Erscheinungen*, Berlin (1898).

§ *Amsterdam Proc.*, Vol. 8 (1900), p. 603.

‖ *Jahresber. Deutsch. Math. Verein*, Vol. 14 (1905), p. 578; *Phys. Zeitschr.*, Vol. 6, p. 803.

¶ *Proc. Camb. Phil. Soc.*, Vol. 15, p. 65.

** *Proc. Amer. Acad. of Arts and Sciences*, Vol. 47 (1911), p. 561.

the charges is not equally divided between the two singular curves, the above argument fails.

It should be noticed that two charges which fluctuate periodically and are connected by only *one* singular curve along which there is a cancelling of singularities, can fluctuate with different periods ; for now we have only a single relation of type

$$f(\tau) = \phi(t),$$

and, unless the two charges are at a constant distance apart, there is no reason why the periods of f and ϕ should be equal.

A NEW METHOD OF DESCRIBING A THREE-BAR CURVE

By Col. R. L. HIPPISLEY, C.B.

[Received November 19th, 1917.—Read December 6th, 1917.]

1. The pedal triangle of a point on a three-bar curve, with reference to the triangle of foci, has the property that its vertices are at fixed distances from a variable point.

This can easily be proved. If α, β, and γ are the angles subtended at the moving point by the sides of such a triangle,

$$\alpha+\beta+\gamma = 2\pi,$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma = 1.$$

Denote the sides of the triangle by x, y, z, and the distances of the vertices from the moving point by r_1, r_2, r_3; then

$$\cos\alpha = \frac{r_3^2+r_1^2-x^2}{2r_3r_1}, \qquad \cos\beta = \frac{r_1^2+r_2^2-y^2}{2r_1r_2}, \qquad \cos\gamma = \frac{r_2^2+r_3^2-z^2}{2r_2r_3};$$

and therefore

$$r_2^2(x^2-r_3^2-r_1^2)^2+r_3^2(y^2-r_1^2-r_2^2)^2+r_1^2(z^2-r_2^2-r_3^2)^2 + (x^2-r_3^2-r_1^2)(y^2-r_1^2-r_2^2)(z^2-r_2^2-r_3^2) = 4r_1^2r_2^2r_3^2. \quad (1)$$

If we denote the focal distances of a point on a three-bar curve by u, v, w, the equation to the curve can be written

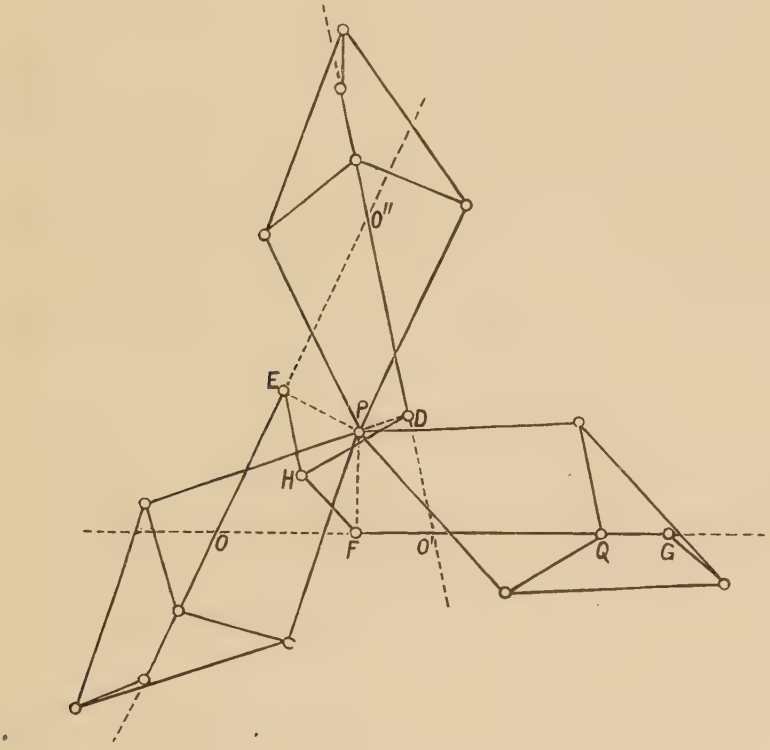
$$a_2^2(u^2-b_1^2-c_3^2)^2+b_3^2(v^2-c_2^2-a_1^2)^2+c_1^2(w^2-a_3^2-b_2^2)^2 + (u^2-b_1^2-c_3^2)(v^2-c_2^2-a_1^2)(w^2-a_3^2-b_2^2) = 4a_1^2b_2^2c_3^2, \quad (2)$$

where a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are the sides of the three traversing triangles.

If we make $r_1 = a_1\sin B$, $r_2 = b_2\sin C$, $r_3 = c_3\sin A$, *i.e.* the perpendicular distances of the point on the curve from the three traversing links, and write for x, y, z the values $u\sin A$, $v\sin B$, $w\sin C$, which

are the lengths of the sides of the pedal triangle for a point (u, v, w), then equation (1) becomes identical with equation (2), showing that such a triangle can be formed for any point on a three-bar curve.

2. This principle may be used for the purpose of constructing a mechanism which will trace out a three-bar curve. If D, E, F are three points, one on each of the sides of the focal triangle $OO'O''$, and we connect them to a variable point H by three bars of lengths l, m, n, then, if they are the vertices of a pedal triangle at all, they will be those of the pedal triangle of a point on a three-bar curve. The only condition that they must observe in order that they should form a pedal triangle is that the perpendiculars through them to the sides of the focal triangle should meet in a point. This can be ensured in the following manner (see figure).



Let the rod FG be made to slide along the side OO'. Attach to it a Peaucellier cell PQG whose fixed pivots are at Q and G, and which is arranged so that the point P describes a straight line through F, perpend-

icular to FG. Apply a similar mechanism to the other two sides of the focal triangle $OO'O''$, and pivot all three points P together; then P describes a three-bar curve. The links l, m, n may have any lengths, provided that any two are not less than the third, and the focal triangle may have any form. The moving links of the three-bar linkage will be respectively

$$\frac{m}{\sin A}, \quad \frac{n \sin C}{\sin A \sin B}, \quad \frac{l}{\sin B},$$

and the fixed side must be less than the sum of these quantities.

It is recognised that the sliding of the bars FG, &c. between guides would prevent the mechanism being a true linkage. But the rod FG can be made to follow the line OO' by the application to it of two Peaucellier cells both designed to trace out the same straight line OO'. It has not been thought necessary to show these cells in the figure.

3. The equation to the locus of the point H can easily be found.

Let x_0, y_0 be the coordinates of P, and x, y those of H. Take the origin at O and OO' as the axis of x. The point H is the intersection of the three circles described with centres D, E, F and radii $b_2 \sin C$, $c_3 \sin A$, and $a_1 \sin B$. The coordinates of D are

$$c_0 - \{(c_0 - x_0)\cos B + y_0 \sin B\} \cos B, \quad \{(c_0 - x_0)\cos B + y_0 \sin B\} \sin B,$$

those of E are

$$(x_0 \cos A + y_0 \sin A)\cos A, \quad (x_0 \cos A + y_0 \sin A)\sin A,$$

and those of F $\qquad x_0, \quad 0,$

where $\qquad OO' = c_0.$

The D circle is

$$\begin{aligned} x^2 + y^2 - 2x\,[c_0 - \{(c_0 - x_0)\cos B + y_0 \sin B\}\cos B] \\ - 2y\,[(c_0 - x_0)\cos B + y_0 \sin B]\sin B \\ + (x_0 \cos B - y_0 \sin B)^2 + (c_0^2 - c_2^2)\sin^2 B = 0. \end{aligned}$$

The E circle is

$$\begin{aligned} x^2 + y^2 - 2x\,(x_0 \cos A + y_0 \sin A)\cos A - 2y\,(x_0 \cos A + y_0 \sin A)\sin A \\ + (x_0 \cos A + y_0 \sin A)^2 - c_3^2 \sin^2 A = 0, \end{aligned}$$

and the F circle is

$$x^2 + y^2 - 2x_0 x + x_0^2 - b_1^2 \sin^2 A = 0.$$

The last circle cuts the first in the straight line

$$2x\,(x_0-c_0+y_0\cot B)+2y\,\{(x_0-c_0)\cot B-y_0\}$$
$$-x_0^2-2x_0y_0\cot B+y_0^2+c_0^2-\mu^2=0, \qquad (3)$$

and the second in the straight line

$$2x\,(x_0-y_0\cot A)-2y\,(x_0\cot A+y_0)-x_0^2+2x_0y_0\cot A+y_0^2-\lambda^2=0, \quad (4)$$

in which $$\lambda^2\equiv c_3^2-b_1^2, \qquad \mu^2\equiv c_2^2-a_1^2.$$

From these two linear equations x and y can be obtained. But we require x_0 and y_0 as functions of x, y. Equations (3) and (4) are

$$(x_0-x)^2+2\,(x_0-x)(y_0-y)\cot B-(y_0-y)^2$$
$$-\{(x-c_0)^2+2\,(x-c_0)\,y\cot B-y^2-\mu^2\}=0,$$
$$(x_0-x)^2-2\,(x_0-x)(y_0-y)\cot A-(y_0-y)^2$$
$$-(x^2-2xy\cot A-y^2-\lambda^2)=0.$$

That is to say, considering x_0 and y_0 as running coordinates, they are both rectangular hyperbolas centred at H, and will have two real and two imaginary intersections. The first is inclined at an angle of $45^\circ-\frac{1}{2}B$, and has a semiaxis

$$\{-(x-c_0)^2\sin B-2\,(x-c_0)\,y\cos B+y^2\sin B+\mu^2\sin B\}^{\frac{1}{2}},$$

and the second is inclined at an angle of $45^\circ+\frac{1}{2}A$, and has a semiaxis

$$(-x^2\sin A+2xy\cos A+y^2\sin A+\lambda^2\sin A)^{\frac{1}{2}}.$$

Writing x', y' for x_0-x, y_0-y, the equations to the two hyperbolas referred to their centres are

$$x'^2+2x'y'\cot B-y'^2+C_1=0,$$
$$x'^2-2x'y'\cot A-y'^2+C_2=0,$$

where $$C_1=-\{(x-c_0)^2+2\,(x-c_0)\,y\cot B-y^2-\mu^2\},$$
$$C_2=-(x^2-2xy\cot A-y^2-\lambda^2)\,;$$

and their intersections are given by

$$x'^2=\frac{-Q\pm\sqrt{P}}{D^2},$$

$$y'^2=\frac{Q\pm\sqrt{P}}{D^2},$$

where

$$Q = C_1 \cot A + C_2 \cot B,$$

$$P = (C_1 - C_2)^2 + (C_1 \cot A + C_2 \cot B)^2,$$

$$D^2 = 2(\cot A + \cot B).$$

We see that P is essentially positive and that $\sqrt{P}$ is greater than Q. Hence we find

$$x_0 = x + \frac{\sqrt{-Q+\sqrt{P}}}{D},\ x - \frac{\sqrt{-Q+\sqrt{P}}}{D},\ x + i\frac{\sqrt{Q+\sqrt{P}}}{D},\ x - i\frac{\sqrt{Q+\sqrt{P}}}{D},$$

$$y_0 = y + \frac{\sqrt{Q+\sqrt{P}}}{D},\ y - \frac{\sqrt{Q+\sqrt{P}}}{D},\ y + i\frac{\sqrt{-Q+\sqrt{P}}}{D},\ y - i\frac{\sqrt{-Q+\sqrt{P}}}{D},$$

and, substituting these values of x_0 and y_0 in the equation to the three-bar curve, we obtain the factors of which the equation to the locus of H is made up. When rationalized it is of the 24th degree.

Since there are two real values of x_0, y_0 for each value of x, y, we see that two real points on the three-bar curve have the same H. Consequently, the point H traverses its orbit twice for each complete circuit of the point P. The equation of the 24th degree may therefore be a perfect square and reducible to one of the 12th degree. The two real values of x_0 and y_0 show that the two points P and P' on the three-bar curve which have a common point H are on the same straight line through H and equidistant from H, the length of HP being given by

$$HP^2 = \frac{2\sqrt{P}}{D},$$

and the inclination of the line by

$$\tan 2\theta = -\frac{C_1 - C_2}{C_1 \cot A + C_2 \cot B}.$$

ON THE CONNEXION BETWEEN LEGENDRE SERIES AND FOURIER SERIES

By Prof. W. H. Young.

[Received November 1st, 1917.—Read December 6th, 1917.]

1. The results hitherto obtained in the theory of the convergence of a Legendre series, defined as a series of the form

$$\Sigma a_n P_n(x), \tag{1}$$

where

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)\, dx, \tag{2}$$

$P_n(x)$ being Legendre's n-th coefficient, namely the coefficient of a^n in the expansion of $(1-2ax+a^2)^{-\frac{1}{2}}$ in powers of x, are somewhat isolated; and a theory of the convergence of a series of these polynomials of the form (1), which does not satisfy (2), series which are *not* Legendre series,* has not, as far as I am aware, been developed at all.

In the present memoir I propose to illustrate a method leading to results which not only include as more or less special corollaries all the theorems already known on the subject of the convergence of these and kindred series, but, at the same time, places their theory on a footing as firm as that of trigonometrical series, at least in so far as the *internal* points of the interval considered are concerned. The end-points of the interval (-1 and 1 in the case of Legendre series) being, in general, singular, require, of course, a special treatment varying with the particular type of series, and in the present communication I have accordingly not introduced them, reserving their discussion for a subsequent paper.

Our method owes its power to the introduction of what I have called Restricted Fourier Series, and to the fundamental theorems which I have been able to establish with regard to them, theorems, the most striking of which I have quite recently communicated to the Society. We are thus enabled to pass from trigonometric series of the restricted Fourier form

* We distinguish between a *Legendre series* and a *series of Legendre coefficients*.

to Legendre series and other series of an extremely general type, with almost the same ease as it is known we can pass to Sturm-Liouville series from ordinary Fourier series. This facility must not be judged by the length at which the present paper has been written out. The reasoning is indeed intuitive in character and the conclusions become almost self-evident when it has been properly grasped.

The great advantage of Restricted Fourier Series, and more especially of what I called *ordinary* Restricted Fourier Series, or R. F. series, is that, in the interval, or intervals, to which we restrict them, they behave more or less like Fourier series. The convergence properties of R. F. series in particular are practically the same as those of Fourier series.

Our method then has the advantage of allowing the student who is acquainted with the theory of Fourier series to see the almost intuitive character of the theorems, and to memorise them without separate enunciation. The use of elaborate analysis, such as has been so ingeniously employed hitherto, is rendered unnecessary, equally whether the convergence required is ordinary, or uniform, or belongs to one of the scales of Cesàro summations, with index integral or fractional. And the series thus rendered manageable are more general in type than those yet considered. They include, for example, not only all convergent Legendre series, but every convergent series of Legendre coefficients whose integrated series converges to an integral in some interval internal to $(-1 < x < 1)$ and containing the point or points under consideration. But further, the nature of the reasoning is such that the present theory may be regarded as fairly typical of the general theory of higher series, whether of normal functions or of more general functions with trigonometric asymptotic approximations. The whole fabric of the theory of these series, with their important applications to mathematical physics, is thus made to repose on the Theory of Fourier Series.

The main theorem obtained in the present paper is as follows :—

At any internal point, or in any internal closed interval, of the interval $(-1 < x < 1)$, *the series of Legendre* (1) *behaves in respect of convergence, divergence, or oscillation, uniform or otherwise, precisely like the Fourier series of* $f(\cos\theta)$, *where*

$$x = \cos\theta,$$

provided the individual terms of the series (1) *tend to zero, that is to say, provided*

$$n^{-\frac{1}{2}} a_n \to 0 \quad (n \to \infty). \tag{3}$$

The theorem still holds for a series of Legendre's coefficients of the form

(1), *provided the first integrated series converge to an integral in some interval containing the point, or points, under consideration, or, what is the same thing, if the sum of the second integrated series [necessarily convergent in virtue of* (3)*] is in such an interval the integral of an integral.**

This latter class of series, as is shewn in § 13, includes; as a special case, Legendre series for which (3) holds.

It follows in particular from this theorem that the upper and lower functions of a Legendre series (1), satisfying (3), as well as (2), at an internal point of the interval $(-1, 1)$, depend only on the values of $f(x)$ in a neighbourhood of the point considered as small as we please, and are given as the upper and lower limits, as $n \to \infty$, of the expression

$$\tfrac{1}{2}(n+1)\int_{x-e}^{x+e}\{P_{n+1}(x)\,P_n(t)-P_n(x)\,P_{n+1}(t)\}\,f(t)\,\frac{dt}{x-t},$$

e being any fixed quantity, as small as we please.

In an Appendix, I give an independent proof of this theorem based on different and more elementary considerations.

In basing our theory on the main theorem of the present paper, we have, as it will be observed, rendered it possible to incorporate, without further analysis, in the theory of our series, any convergence criterion for Fourier series which may in the future be discovered. Applying those tests which are at present known, we have, moreover, at once a number of results which need not be enumerated here. I may, however, perhaps remind the reader that all the known tests are summed up in the two given respectively by de la Vallée Poussin and myself :—

(1) $\dfrac{1}{2u}\displaystyle\int_0^u \{f(x+u)+f(x-u)\}\,du$ *is a function of bounded variation*; *the Fourier series of* $f(x)$ *then converges to*

$$\operatorname*{Lt}_{u=0}\frac{1}{2u}\int_0^u\{f(x+u)+f(x-u)\}\,du:$$

and

(2) $$\tfrac{1}{2}\{f(x+u)+f(x-u)\}\to C,\quad (u\to 0),$$

while

$$\frac{1}{u}\int|du\,\{f(x+u)+f(x-u)\}|$$

is bounded; *the Fourier series of* $f(x)$ *then converges to* C.

* The special case of the theorem dealing with *the Legendre series of a function whose square is summable* had been proved by quite different methods, adapted *ad hoc*, by A. Haar (*Math. Annalen*, Vol. 78, 1918, pp. 121–136).

It follows then from the results of the present paper that, *under the same circumstances, the Legendre series of $f(x)$ converges, at any point x internal to $(-1 < x < 1)$, to the same value as the Fourier series, and that the difference of corresponding partial summations tends uniformly to zero.* We thus get tests for the convergence for a Legendre series, neither of which has as yet been established for any but Fourier series and R. F. series.

I have not thought it wise to lengthen the paper by dealing explicitly with Cesàro convergence. No fresh difficulty is introduced, if we do so, except that the corresponding theorems regarding the Cesàro convergence of Restricted Fourier Series would be required, and that a closer asymptotic formula, the theory of which may now be regarded as fairly complete, would need to be applied for the higher indices.

It has already been remarked that the method here used is fruitful in connexion with series of many other types. There are special reasons, however, for considering Legendre series first and separately. After Fourier series, they are perhaps the best known of all series of real functions, and are calculated to interest a large class of readers. Moreover, in their treatment the use of the complex variable is entirely dispensed with, whereas, in applying the method to other expansions, more or less well known, it is often convenient to employ the method of residues for the expansion of a suitable power of the independent variable, so as to be able, as in § 3, to apply Lerch's theorem. I propose to discuss elsewhere some of the series here contemplated.

2. We shall require in the sequel the following simple theorem in Fourier series :—

THEOREM.—*If at the point x the derivates of $g(x)$ are all finite, the upper and lower functions of the Fourier series of $f(x)\,g(x)$, where $f(x)$ is any summable function, are got from those of the Fourier series of $f(x)$ by multiplying by $g(x)$.*

In fact, ϵ being as small as is convenient, we have, by the usual formula for the partial summations of the Fourier series of $f(x)$ at the origin, which for convenience we take to be the point under consideration,

$$s_n = \frac{1}{2\pi}\int_0^\epsilon \{f(u)+f(-u)\} \sin(n+\tfrac{1}{2})\,u \operatorname{cosec} \tfrac{1}{2}u\,du,$$

and S_n, the corresponding partial summation for $f(x)\,g(x)$, is got by replacing $\{f(u)+f(-u)\}$ by $f(u)\,g(u)+f(-u)\,g(-u)$.

Now

$$\frac{1}{2\pi}\int_0^\epsilon f(u)\,g(u)\sin(n+\tfrac{1}{2})u\,\operatorname{cosec}\tfrac{1}{2}u\,du$$

$$= \frac{g(0)}{2\pi}\int_0^\epsilon f(u)\sin(n+\tfrac{1}{2})u\,\operatorname{cosec}\tfrac{1}{2}u\,du$$

$$+\frac{1}{2\pi}\int_0^\epsilon \frac{g(u)-g(0)}{u}\,u\,\operatorname{cosec}\tfrac{1}{2}u\,f(u)\sin(n+\tfrac{1}{2})u\,du.$$

Here, ϵ being chosen conveniently small, $\{g(u)-g(0)\}/u$ is bounded, since, by hypothesis, the derivates of g at the origin are all finite. Hence the factor of $\sin(n+\frac{1}{2})u$ in the integrand of the second integral is a summable function. Therefore, by the theorem of Riemann-Lebesgue, this integral tends to zero as $n\to\infty$.

Similarly for the other term of S_n. Thus, adding

$$S_n = g(0)\,s_n+\eta_n,$$

where $\eta_n\to 0$ as $n\to\infty$. This proves the theorem.

Cor.—*The difference between the corresponding partial summations of the Fourier series of $g(x)f(x)$ and of $f(x)$, when the latter partial summation is multiplied by $g(x)$, converges uniformly to zero.*

In fact, as in the above proof, we get

$$S_n(x)-g(x)\,s_n(x) = \eta_n(x),$$

where, by Lebesgue's addition to the theorem already quoted, $\eta_n(x)$ tends *uniformly* to zero as $n\to\infty$.

3. We shall next establish certain simple results required in the proof of the main theorem:—

Lemma 1.—*If $h(x)$ is a summable function, and it is known that, for all integers n,*

$$\int_{-1}^{1} h(x)\,P_n(x)\,dx = 0,$$

then $h(x) = 0$ almost everywhere in $(-1, 1)$.

For $$x^n = a_0P_n+a_1P_{n-1}+\ldots+a_nP_0,$$

$a_0, a_1, \ldots$ being constants. Therefore, for all integers n,

$$\int_{-1}^{1} x^n h(x)dx = 0;$$

therefore by the extended theorem of Lerch* the result follows.

4. LEMMA 2.—*The second integrated series of the series of Legendre*

$$f(x) \sim \sum_{k=0}^{\infty} a_n P_n(x), \tag{1}$$

which satisfies the necessary convergence condition

$$n^{-\frac{1}{2}} a_n \to 0 \quad (n \to \infty), \tag{3}$$

converges uniformly to $\int_a^x dx \int_a^x f(x)\,dx$.

To prove this we use the formula, which characterises (1) as a Legendre series,

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)dx, \tag{2}$$

and the familiar identity

$$(2n+1)\, P_n = \frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx}. \tag{4}$$

We then have, writing $F(x) = \int f(x)\,dx$,

$$A_n = (n+\tfrac{1}{2}) \int_{-1}^{1} F(x)\, P_n(x)dx,$$

$$\begin{aligned}\frac{a_{n-1}}{2n-1} - \frac{a_{n+1}}{2n+3} &= -\tfrac{1}{2}\int_{-1}^{1} f(x)(P_{n+1}-P_{n-1})\,dx \\ &= \tfrac{1}{2}\Big[F(x)(P_{n-1}-P_{n+1})\Big]_{-1}^{1} + \tfrac{1}{2}\int_{-1}^{1} F(x)\left(\frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx}\right)dx \\ &= A_n. \end{aligned} \tag{5}$$

* This extension appears to have been first given in my paper, "On a Class of Symmetric Functions and on a Theorem required in the Theory of Integral Equations," *Messenger of Mathematics*, 1910, Vol. 40, pp. 37–43. It was re-proved by Hobson (*Proc. London Math. Soc.*, Ser. 2, Vol. 11) who gives references to Lerch and Landau for the continuous case.

It is convenient to use the theorem habitually in its extended form, though in the application below in the text (§ 4) the theorem in its original form (Lerch's) would be sufficient.

Hence, by (3), $$A_n\sqrt{n}\to 0 \quad (n\to\infty). \qquad (6)$$

Also $$-\frac{a_1}{3}=-\tfrac{1}{2}\int_{-1}^{1} f(x)\,P_1(x)\,dx=\tfrac{1}{2}\{F(1)+F(-1)\}+A_0. \qquad (7)$$

Hence, using the identity got by integrating (4),

$$\sum_{n=0}^{m}\int a_nP_n(x)\,dx=a_0P_1+\sum_{n=1}^{m}a_n\frac{P_{n+1}-P_{n-1}}{2n+1}$$

$$=-\frac{a_1}{3}P_0+\sum_{n=1}^{m-1}\left(\frac{a_{n-1}}{2n-1}-\frac{a_{n+1}}{2n+3}\right)P_n+\frac{a_{n-1}P_n}{2n-1}+\frac{a_nP_{n+1}}{2n+1}$$

$$=\tfrac{1}{2}\{F(1)+F(-1)\}+\sum_{n=1}^{m-1}A_nP_n+\frac{a_{n-1}P_n}{2n-1}+\frac{a_nP_{n+1}}{2n+1}. \qquad (8)$$

Writing $$G(x)=\int F(x)\,dx, \qquad B_n=(n+\tfrac{1}{2})\int_{-1}^{1}G(x)\,P_n(x)\,dx,$$

we can integrate $\sum_{n=1}^{m-1}A_nP_n$, as we did $\sum_{n=1}^{m}a_nP_n$, and thus get, without further calculation,

$$\sum_{n=0}^{m}\int dx\int a_nP_n(x)\,dx=\tfrac{1}{2}\{F(1)+F(-1)\}\,x+\tfrac{1}{2}\{G(1)+G(-1)\}+\sum_{n=1}^{m-2}B_nP_n$$

$$+\frac{A_{n-2}P_{n-1}}{2n-3}+\frac{A_{n-1}P_n}{2n}+\frac{a_{n-1}(P_{n+1}-P_{n-1})}{(2n-1)(2n+1)}+\frac{a_n(P_{n+2}-P_n)}{(2n+1)(2n+3)}$$

$$=\tfrac{1}{2}\{F(1)+F(-1)\}\,x+\tfrac{1}{2}\{G(1)+G(-1\}+\sum_{n=1}^{m-2}B_nP_n+O(n^{-\frac{3}{2}}), \qquad (9)$$

using (3) and (5). But

$$B_n=\frac{A_{n-1}}{2n-1}-\frac{A_{n+1}}{2n+3}=O(n^{-\frac{3}{2}}). \qquad (10)$$

Therefore $\sum_{n=1}^{\infty}B_nP_n$ converges uniformly, like $\Sigma n^{-\frac{3}{2}}$. Denote the sum of this series by $\phi(x)$, so that

$$\phi(x)=\sum_{n=1}^{\infty}B_nP_n.$$

Now multiply by $P_n(x)$ and integrate term by term, and we get

$$\int_{-1}^{1}\phi(x)\,P_n(x)\,dx=B_n.$$

Therefore $$\int_{-1}^{1}\{\phi(x)-G(x)\}\,P_n(x)\,dx=0,$$

for all integers n. Hence, by Lemma 1,

$$\phi(x) \equiv G(x).$$

That is,
$$G(x) = \sum_{n=1}^{\infty} B_n P_n. \tag{11}$$

Hence, by (9), letting $m \to \infty$,

$$\sum_{n=0}^{\infty} \int dx \int a_n P_n(x)\,dx = \tfrac{1}{2}\{F(1)+F(-1)\}\,x + \tfrac{1}{2}\{G(1)+G(-1)\} + G(x). \tag{12}$$

Choosing the integral $F(x)$ suitably $F(1)+F(-1)=0$, and choosing the integral $G(x)$ suitably $G(1)+G(-1)=0$, and we have the required result

$$\sum_{n=0}^{\infty} \int dx \int a_n P_n(x)\,dx = G(x) = \int dx \int f(x)\,dx. \tag{13}$$

5. In the theory of term-by-term integration, one of the most familiar theorems is perhaps that which states that, if $\Sigma u_n(x)$ converges uniformly to $f(x)$, we may multiply by any *summable function* $g(x)$ and integrate term-by-term, that is

$$\Sigma \int g(x)\,u_n(x)\,dx = \int g(x) f(x)\,dx.$$

More generally, we may integrate with respect to any *function* $h(x)$ *of bounded variation*, and write

$$\Sigma \int u_n(x)\,dh(x) = \int f(x)\,dh(x).$$

Again, if $\Sigma \int u_n(x)\,dx$ converges uniformly to $\int f(x)\,dx$, we may multiply the series $\Sigma u_n(x)$ by any *function of bounded variation* $g(x)$ and integrate term-by-term,* that is as before

$$\Sigma \int g(x)\,u_n(x)\,dx = \int g(x) f(x)\,dx,$$

the new series still converging uniformly.

In the present connexion we require to go a step further, and enunciate accordingly the following Lemma.

LEMMA 3.—*If* $\Sigma \int dx \int u_n(x)\,dx$ *converges uniformly to* $\int dx \int f(x)\,dx$, *then*

* See my paper "On the Application of Expansions to Definite Integrals" (1910), § 10, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 475.

$\Sigma \int \mu(x) \{\int \lambda(x) u_n(x)\, dx\}\, dx$ *converges uniformly to* $\int \mu(x) \{\int \lambda(x) f(x)\, dx\}\, dx$, *provided* $d\lambda/dx$ *and* $\mu(x)$ *are functions of bounded variation.*

Write $$\int_a^x f(x)\,dx = F(x), \qquad \int_a^x F(x)\,dx = G(x),$$

$$\int_a^x u_n(x)\,dx = U_n(x), \qquad \int_a^x U_n(x)\,dx = V_n(x)$$

Then, by hypothesis, $\Sigma V_n(x)$ converges to $G(x)$ uniformly.

Now, integrating by parts twice, we have

$$\int_a^y \lambda(x)\, u_n(x)\, dx = \lambda(y)\, U_n(y) - \lambda'(y)\, V_n(y) + \int_a^y V_n(x)\, d\lambda'(x).$$

Hence

$$\Sigma \int_a^z \mu(y) \left\{\int_a^y \lambda(x)\, u_n(x)\, dx\right\} dy$$

$$= \Sigma \left[\int_a^z \mu(y)\,\lambda(y)\, U_n(y)\, dy - \int_a^z \mu(y)\,\lambda'(y)\, V_n(y)\, dy \right.$$

$$\left. + \int_a^z \mu(y) \left\{\int_a^y V_n(x)\, d\lambda'(x)\right\} dy\right]$$

$$= \int_a^z \mu(y)\,\lambda(y)\, F(y)\, dy - \int_a^z \mu(y)\,\lambda'(y)\, G(y)\, dy + \int_a^z \mu(y) \int_a^y G(x)\, d\lambda'(x)\, dy,$$

since, λ' and μ being functions of bounded variation, $\mu\lambda$ and $\mu\lambda'$ are functions of bounded variation (and therefore summable), so that the above result follows at once from the theorems quoted.

But, integrating by parts twice again, we see that $\int_a^y \lambda(x)\, f(x)\, dx$ multiplied by $\mu(y)$ and integrated is the right-hand side of the last equation.

This therefore proves the Lemma.

In particular of course *the Lemma holds when* $\lambda(x)$ *and* $\mu(x)$ *are functions possessing their first three differential coefficients.*

6. From Lemmas 2 and 3 we at once have the following:—

Lemma 4.—*If* $$f(x) \sim \sum_{n=1}^{\infty} a_n P_n(x),$$

be a Legendre series such that $n^{-\frac{1}{2}} a_n \to 0$, *the series*

$$\sum_{k=}^{\infty} \int_a^z dz \int_a^z \sqrt{(\sin^3 2z)}\, a_n P_n(\cos 2z)\, dz,$$

converges uniformly at every point, or in any closed interval in the completely open interval $(0 < z < \frac{1}{2}\pi)$, *to*

$$\int_a^z dz \int_a^z \sqrt{(\sin^3 2z)}\, f(\cos 2z)\, dz.$$

In fact, writing $x = \cos 2z$, $\cos \alpha = a$, where $0 < \alpha < \beta \leqslant \frac{1}{2}\pi$,

$$\int_a^z dz \int_a^z \sqrt{(\sin^3 2z)}\, a_n P_n(\cos 2z)\, dz = \tfrac{1}{4} \int_a^x (1-x^2)^{-\frac{1}{2}} dx \int_a^x (1-x^2)^{\frac{1}{2}}\, a_n P_n(x)\, dx.$$

By Lemmas 1 and 2, this is the generic term of a series which converges uniformly to

$$\tfrac{1}{4} \int_a^x (1-x^2)^{-\frac{1}{2}}\, dx \int_a^x (1-x^2)^{\frac{1}{2}} f(x)\, dx = \int_a^z dz \int_a^z \sqrt{(\sin^3 2z)}\, f(\cos 2z)\, dz,$$

since the functions $(1-x^2)^{-\frac{1}{2}}$ and $(1-x^2)^{\frac{1}{2}}$ possess all their repeated differential coefficients in the closed interval considered.

This proves the Lemma.

7. We can now prove the main theorem :—

THEOREM.—*The series of Legendre*

$$\sum_{k=0}^{\infty} a_n P_n(x) \equiv \sum_{k=0}^{\infty} a_n P_n(\cos\theta), \tag{1}$$

where $x = \cos\theta$, *and* $a_n = (n+\frac{1}{2}) \int_{-1}^{1} f(t)\, P_n(t)\, dt$, (2)

behaves in respect of convergence, divergence, and oscillation, uniform or non-uniform, at a point, or in a closed interval, inside the completely open interval $(0 < \theta < \pi)$, *that is* $(-1 < x < 1)$, *precisely like the Fourier series of* $f(\cos\theta)$, *provided*

$$n^{-\frac{1}{2}} a_n \to 0 \quad (n \to \infty). \tag{3}$$

To prove this theorem we use the asymptotic expression for P_n given by Heine.* This may be written as follows :—

$$P_n(\cos\theta) = Q_n(\theta) + O(n^{-\frac{1}{2}}), \tag{4}$$

* The formula is well-known and easily proved. Heine's proof is perhaps not quite satisfactory : other proofs have been given.

where

$$Q_n(\theta) = \sqrt{\left(\frac{2}{n\pi \sin\theta}\right)}\left[\left(1-\frac{1}{4n}+\frac{A}{n^2}\right)\cos\omega_n+\left(\frac{1}{8n}+\frac{B}{n^2}\right)\cot\theta\sin\omega_n\right]$$
$$+C(n\sin\theta)^{-\frac{5}{2}}\cos\omega_n, \quad (5)$$

A, B, and C being numerical constants (independent of n), and

$$\omega_n = (n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi. \quad (6)$$

Thus since $2\cos\omega_n\sin\theta = -\sin\omega_{n-1}+\sin\omega_{n+1}$,

and $2\sin\omega_n\cos\theta = \sin\omega_{n-1}+\sin\omega_{n+1}$,

we have, writing

$$\lambda_n = \frac{1}{\sqrt{(2n\pi)}}\left(1-\frac{3}{8n}+\frac{A-B}{n^2}\right), \quad \mu_n = \frac{1}{\sqrt{(2n\pi)}}\left(1-\frac{1}{8n}+\frac{A+B}{n^2}\right), \quad (7)$$

$$\sqrt{(\sin^3\theta)}\,Q_n(\theta)-Cn^{-\frac{5}{2}}\operatorname{cosec}\theta\cos\omega_n = -\lambda_n\sin\omega_{n-1}+\mu_n\sin\omega_{n+1}.$$

Hence

$$\sqrt{(\sin^3\theta)}\sum_{n=1}^{m}a_nQ_n(\theta)-C\operatorname{cosec}\theta\sum_{n=1}^{m}a_nn^{-\frac{5}{2}}\cos\omega_n$$
$$= -a_1\lambda_1\sin\omega_0-a_2\lambda_2\sin\omega_1+\sum_{n=2}^{m}(-a_{n+1}\lambda_{n+1}+a_{n-1}\mu_{n-1})\sin\omega_n$$
$$+a_{m+1}\lambda_{m+1}\sin\omega_m+a_m\mu_m\sin\omega_{m-1},$$

where, by (7), the last two terms are numerically less than

$$\frac{|a_{m+1}|}{\sqrt{[2(m+1)\pi]}}, \quad \frac{|a_m|}{\sqrt{(2m\pi)}}$$

respectively, provided m is sufficiently large.

Thus, putting $z = \frac{1}{2}\theta$, and writing the series on the right of the last equation in the form of a trigonometrical series in $\cos nz$ and $\sin nz$ (the coefficients of the even terms will then, of course, all be zero), we get*

$$\sqrt{(\sin^3 2z)}\sum_{n=1}^{m}a_nQ_n(\theta)-C\operatorname{cosec}2z\sum_{n=1}^{m}a_nn^{-\frac{5}{2}}\cos\omega_n$$
$$= \sum_{n=1}^{m}(A_n\cos nz+B_n\sin nz)+\epsilon_m, \quad (8)$$

* It should be remarked that we may if we please, in the expression for Q_n, multiply up by $(\sin\theta)^{\frac{5}{2}}$ instead of $(\sin\theta)^{\frac{3}{2}}$, and so absorb the term $Cn^{-\frac{5}{2}}\operatorname{cosec}\theta\cos\omega_n$. The expression we ob-

where $\epsilon_m \to 0$ when $m \to \infty$, by (3), and is numerically less than the absolute value of $\{m^{-\frac{1}{2}} a_m + (m+1)^{-\frac{1}{2}} a_{m+1}\}/\sqrt{(2\pi)}$.

We may write the last relation in the form

$$\sqrt{(\sin^3 2z)} \sum_{n=1}^{\infty} a_n Q_n(2z) - C \operatorname{cosec} 2z \sum_{n=1}^{\infty} a_n n^{-\frac{5}{2}} \cos \omega_n = \sum_{n=1}^{\infty} (A_n \cos nz + B_n \sin nz), \quad (9)$$

remarking, however, that this relation must not be understood to imply convergence of the series involved. It is a *symbolic* equation, and means that, *apart from an additive term of the order* $m^{-\frac{1}{2}} a_m$, *which, by* (3), *tends to zero uniformly*, the equation is actually true, when for infinity we substitute m.

We see, by (7) and (3), that

$$A_n \to 0, \quad B_n \to 0 \quad (n \to \infty). \quad (10)$$

8. We shall now prove that *the series*

$$\sum_{n=1}^{\infty} (A_n \cos nz + B_n \sin nz), \quad (11)$$

is an R. F. series in the interval $(0 < z < \frac{1}{2}\pi)$, and find its associated function $g(z)$.

Remembering (3), we see by (4) that the continuous function $a_n(P_n - Q_n)$ is the generic term of a uniformly convergent series. Similarly, $a_n n^{-\frac{3}{2}} \cos \omega_n$ is the generic term of another such series. De-

---

tain, viz.

$$2\sqrt{(\sin^5 \theta)}\, Q_n(\theta) = \lambda_n \cos \omega_{n-2} + (\mu_n - \lambda_n + 2Cn^{-\frac{5}{2}}) \cos \omega_n - \mu_n \cos \omega_{n+2},$$

leads then to an equation of the form

$$\sqrt{(\sin^5 \theta)}\, Q_n(\theta) = \sum_{n=1}^{m} (A_n \cos nz + B_n \sin nz) + \epsilon_m,$$

in place of (8), and corresponding changes in the remaining equations.

This simplification has certain advantages and is more especially convenient in dealing with the Cesàro convergence, as it may then save an order in the approximation. Thus, for example, if we take an ordinary Legendre series, for which (3) does not hold, we have necessarily,

$$a_n/n \to 0 \quad (n \to \infty),$$

and the approximation used in the text is then sufficient to shew that, without any further hypothesis as to the order of a_n, the $(C1)$ properties of the Legendre series (1) and the Fourier series of $f(\cos\theta)$ are the same. It should be noticed, however that, in the case of fractional Cesàro summation, the index of summation must be higher by $\frac{1}{2}$ for the Legendre than for the Fourier series.

noting by $\phi_1(2z)$ and $\phi_2(2z)$ the continuous functions which are the sums of these series, so that

$$\phi_1(2z) = \sum_{n=1}^{\infty} a_n (P_n - Q_n), \tag{12}$$

$$\phi_2(2z) = \sum_{n=1}^{\infty} a_n n^{-\frac{3}{2}} \cos \omega_n, \tag{13}$$

we have, by (9) and (4),

$$\sum_{n=1}^{\infty} (A_n \cos nz + B_n \sin nz)$$
$$= \sqrt{(\sin^3 2z)} \sum_{n=1}^{\infty} a_n P_n(\cos 2z) - \sqrt{(\sin^3 2z)}\, \phi_1(2z) - C \operatorname{cosec} 2z\, \phi_2(2z), \tag{14}$$

a relation which is again to be understood not to contain implicitly any assertion of the convergence of the series, but which is true, when for infinity we substitute m, neglecting a term which tends uniformly to zero as $m \to \infty$.

From this symbolic equation we can therefore obtain a veritable equation by integrating twice term by term with respect to z in (α, β), where $0 < \alpha < z < \beta < \frac{1}{2}\pi$, since, by Lemma 4, the right-hand side of (14) then gives us a uniformly convergent series. We thus get, by that Lemma,

$$\sum_{n=1}^{\infty} \int_a^z dz \int_a^z (A_n \cos nz + B_n \sin nz)\, dz$$
$$= \int_a^z dz \int_a^z \left[\sqrt{(\sin^3 2z)}\, \{f(\cos 2z) - \phi_1(2z)\} - C \operatorname{cosec} 2z\, \phi_2(2z)\right] dz. \tag{15}$$

Thus the second integrated series of the series (11) converges to a two-fold integral in the interval (α, β); therefore, since (α, β) was any closed interval in $(0 < z < \frac{1}{2}\pi)$, the series (11) is, by definition, a restricted Fourier series of the second class, and its associated function $g(z)$ is the integrand of the integral on the right of (15).

But, since (10) holds, this restricted Fourier series (11) is *ordinary*, that is, it is an R. F. series in $(0 < z < \frac{1}{2}\pi)$, and its associated function is

$$g(z) = \sqrt{(\sin^3 2z)}\, f(\cos 2z) - \sqrt{(\sin^3 2z)}\, \phi_1(2z) - C \operatorname{cosec} 2z\, \phi_2(2z). \tag{16}$$

By the theory of R. F. series, it follows that the upper and lower functions of the series (10) are the same as those of the Fourier series of $g(z)$, provided $0 < z < \frac{1}{2}\pi$.

9. We shall next shew that *the Fourier series of ϕ_1 and ϕ_2 converge uniformly.* In fact both these functions are integrals.

To prove that ϕ_1 is an integral, we calculate $\frac{d}{d\theta}(P_n - Q_n)$ by means of (5).

Using a familiar identity and the relation (4), we have

$$\frac{dP_n}{d\theta} = n \operatorname{cosec}\theta(\cos\theta P_n - P_{n-1}) = n \operatorname{cosec}\theta(\cos\theta Q_n - Q_{n-1}) + O(n^{-\frac{1}{2}}). \quad (17)$$

Replacing $n-1$ by $n(1-1/n)$, and expanding the powers which occur by the Binomial Theorem to as many terms as is required, we have, by (5),

$$Q_{n-1}(\theta) = \sqrt{\left(\frac{2}{n\pi\sin\theta}\right)}\left[\left\{1+\frac{1}{2n}+\frac{3}{8n^2}-\frac{1}{4n}\left(1+\frac{3}{2n}\right)+\frac{A}{n^2}\right\}\cos\omega_{n-1}\right.$$
$$\left.+\frac{1}{8n}\left(1+\frac{3}{2n}+\frac{8B}{n}\right)\cot\theta\sin\omega_{n-1}\right]$$
$$+C(n\sin\theta)^{-\frac{5}{2}}\cos\omega_{n-1}+O(n^{-\frac{7}{2}});$$

and therefore, subtracting from $\cos\theta Q_n$, as given by (5), and multiplying the result by $n\operatorname{cosec}\theta$,

$$\frac{dP_n}{d\theta} = \sqrt{\left(\frac{2}{n\pi\sin^3\theta}\right)}\left[-\left(n+\tfrac{1}{4}+\frac{A}{n}\right)\sin\theta\sin\omega_n-\tfrac{1}{2}\cos\theta\cos\omega_n\right.$$
$$\left.-\tfrac{3}{16}\cot\theta\cos\theta\sin\omega_n+\left(\tfrac{1}{8}+\frac{3}{16n}+\frac{B}{n}\right)\cos\theta\cos\omega_n\right]$$
$$-Cn^{-\frac{3}{2}}\sqrt{(\operatorname{cosec}^5\theta)}\sin\omega_n+O(n^{-\frac{5}{2}}). \quad (18)$$

Also, differentiating (5), and using the identity

$$\operatorname{cosec}\theta\sin\omega_n = \sin\theta\sin\omega_n+\cot\theta\cos\theta\sin\omega_n,$$

we get

$$\frac{dQ_n}{d\theta} = \sqrt{\left(\frac{2}{n\pi\sin^3\theta}\right)}\left[-\frac{1}{2}\left(1-\frac{1}{4n}\right)\cos\theta\cos\omega_n-\frac{1}{16n}\cot\theta\cos\theta\sin\omega_n\right.$$
$$-\left(1-\frac{1}{4n}+\frac{A}{n^2}\right)(n+\tfrac{1}{2})\sin\theta\sin\omega_n+\left(\frac{1}{8n}+\frac{B}{n^2}\right)(n+\tfrac{1}{2})\cos\theta\cos\omega_n$$
$$\left.-\frac{1}{8n}(\sin\theta\sin\omega_n+\cot\theta\cos\theta\sin\omega_n)\right]$$
$$-Cn^{-\frac{5}{2}}(n+\tfrac{1}{2})\sqrt{(\operatorname{cosec}^5\theta)}\sin\omega_n+O(n^{-\frac{5}{2}}). \quad (19)$$

By (18) and (19) we have at once

$$\frac{dP_n}{d\theta}-\frac{dQ_n}{d\theta} = O(n^{-\frac{3}{2}}). \quad (20)$$

Thus $a_n \frac{d}{d\theta}(P_n - Q_n)$ is the generic term of a uniformly convergent series, and we may, in consequence, differentiate (12) term-by-term, which gives

$$\frac{d\phi_1(\theta)}{d\theta} = \sum_{n=1}^{\infty} a_n \frac{d}{d\theta}(P_n - Q_n),$$

expressed as a uniformly convergent series. Integrating this equation term-by-term, which is accordingly allowable, we can see that $\phi_1(\theta)$ is an integral, which proves our required result.

10. To shew that ϕ_2 is an integral, we merely have to remark that the series

$$\sum_{n=1}^{\infty} a_n n^{-\frac{3}{2}} (n+\tfrac{1}{2}) \sin \omega_n,$$

when expressed as a series in terms of $\cos(2n+1)z$ and $\sin(2n+1)z$, is a Fourier series, by the Riesz-Fischer theorem, since the squares of the coefficients form a convergent series. Thus, by (13), $\phi_2(2z)$ is the sum of the integrated series of a Fourier series, and is accordingly an integral.

11. Thus, as stated in § 9, the Fourier series of $\phi_1(2z)$ and $\phi_2(2z)$ converge uniformly. The same is therefore true, in any interval (α, β), where $0 < \alpha < \beta < \frac{1}{2}\pi$, of those of $\sqrt{(\sin^3 2z)}\,\phi_1(2z)$ and $C \operatorname{cosec} 2z\, \phi_2(2z)$.

But, by (16), the Fourier series of $g(z)$ is the sum of the Fourier series of

$$\sqrt{(\sin^3 2z)}\, f(\cos 2z), \quad -\sqrt{(\sin^3 2z)}\, \phi_1(2z), \quad -C \operatorname{cosec} 2z\, \phi_2(2z),$$

this statement being understood to be true of the partial summations but not to involve convergence of the series. Since, however, the two latter Fourier series have been shewn to converge uniformly to their associated functions, we see that the m-th partial summation of the Fourier series of $g(z)$ differs from that of $\sqrt{(\sin^3 2z)}\, f(\cos 2z)$ by

$$-\sqrt{(\sin^3 2z)}\, \phi_1(2z) - C \operatorname{cosec} 2z\, \phi_2(2z),$$

neglecting a term which tends uniformly to zero as $m \to \infty$.

Hence, by (14), $\sqrt{(\sin^3 2z)} \sum_{n=1}^{m} a_n P_n(\cos 2z)$

differs from the m-th partial summation of the Fourier series of

$$\sqrt{(\sin^3 2z)}\, f(\cos 2z),$$

by a term which tends uniformly to zero.

12. Now, in the interval (α, β), $\sqrt{(\sin^3 2z)}$ has a bounded differential coefficient; therefore, by the Corollary to the Theorem of § 2, the latter partial summation is expressible in the form

$$\sqrt{(\sin^3 2z)}\, s_m(\cos 2z) + \eta_m(z),$$

where s_m is the corresponding partial summation of the Fourier series of $f(\cos 2z)$, and η_m tends uniformly to zero.

Thus, finally, replacing $2z$ by θ,

$$\sum_{n=1}^{m} a_n P_n(\cos\theta) = s_m(\cos\theta) + H_m(\theta),$$

where $H_m(\theta)$ tends uniformly to zero at any point, or in any closed interval, inside $(0 < \theta < \pi)$.

Hence the theorem as enunciated at once follows.

13. The following result, *if assumed known*, might have been used to shorten the above proof that the series (11) is an R. F. series.

COR. 1.—*The first integrated series*

$$\Sigma \int a_n P_n(x)\, dx$$

of a Legendre series for which

$$n^{-\frac{1}{2}} a_n \to 0 \quad (n \to \infty) \tag{3}$$

converges uniformly to an integral at every point and in every interval in $(-1 < x < 1)$.

For we have seen that the left-hand side of (14) is an R. F. series, and accordingly its integrated series converges uniformly to an integral in $(\alpha \leqslant z \leqslant \beta)$.

Hence, integrating (14) term-by-term (which is allowable by reason of the uniform convergence of the left-hand side), after multiplying by $\sqrt{(\operatorname{cosec} 2z)}$, the required result appears immediately.

14. We also have immediately the following extension of our results to a wider class of expansions :—

COR. 2.—*The theorem is still true for a series of polynomials of Legendre of the form* (1), *which satisfies* (3), *without being a series of Legendre* [*that is, without satisfying* (2)], *provided in some interval containing the point or points under consideration the sum of the second integrated series is the integral of an integral.*

In fact the relation (2)—which nowhere enters explicitly into the proof of the above theorem—is only involved *implicitly* in the citation of Lemma 4. It occurs explicitly only in the proof of Lemma 2, involved in that of Lemma 4. The assumption made in the present corollary renders the use of Lemma 4 superfluous. This proves the corollary.

We notice also, that by the same argument as in Cor. 1, *the first integrated series of such a series as we here contemplate converges to an integral.* It will be seen by the next two theorems that *this convergence is uniform*, and that the present corollary is precisely equivalent, in all but form, to the theorem of § 16.

15. We can now prove the following theorem :—

Theorem.—*If a series of Legendre polynomials (not necessarily a Legendre series)*

$$\Sigma a_n P_n(x), \tag{1}$$

for which

$$n^{-\frac{1}{2}} a_n \to 0 \tag{3}$$

converges to an integral in some sub-interval of $(-1, 1)$, *not including either end-point, then it converges uniformly to that integral.*

In fact (14) holds as an actual equation in our interval, and the right-hand side is an integral, since the proof of this relation (14), as well as the proof that ϕ_1 and ϕ_2 are integrals, do not involve the fulfilment of the relation (3).

But a trigonometrical series which converges to an integral is an R. F. series.*

Thus the series on the left of (14) is an R. F. series, and therefore—since it converges to an integral, and behaves like a Fourier series—converges uniformly throughout the sub-interval in question. Hence also the series on the right converges uniformly. This proves the theorem.

16. Hence we have the following extension of the main theorem.

Theorem.—*The main theorem* (§ 7) *remains true for any series of the form* (1), *which satisfies* (3) (*without being a series of Legendre*), *provided*

* For, as pointed out in my paper on "Restricted Fourier Series and the Convergence of Power Series", the series of theorems initiated by Riemann, and continued by de la Vallée Poussin and myself, on certain conditions under which a trigonometrical series is a Fourier series, apply in a sub-interval of the interval of periodicity to shew that under the same circumstances the trigonometrical series is an R. F. series. Among these the simplest is that relating to the case in which the series converges to a continuous or a bounded function.

that in some interval containing the point, or points, under consideration, the first integrated series converges to an integral.

In fact, since

$$\int a_n P_n(x)\,dx = a_n \{P_{n+1}(x) - P_{n-1}(x)\}/(2n+1),$$

the first integrated series of (1) is a series of the form (1) whose coefficients tend to zero. Hence, by the preceding theorem, it converges *uniformly* to the integral which is, by hypothesis, its sum.

We may therefore integrate it term-by-term, and find the conditions of Cor. 1 (§ 14) satisfied.

This proves the theorem, and at the same time shows, as there pointed out, that *this theorem is in all but form precisely equivalent to that corollary* (§ 14).

Appendix.

17. THEOREM.—*If*

$$n^{-\frac{1}{2}} a_n \to 0, \quad (n \to \infty), \tag{3}$$

the upper and lower functions of the Legendre series

$$\sum_{n=0}^{\infty} a_n P_n(x), \tag{1}$$

where

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)\,dx, \tag{2}$$

at any internal point of the interval $(-1, 1)$, *depend only on the values of* $f(x)$ *in a neighbourhood of the point considered, as small as we please.*

A similar statement is true of the peak and chasm functions.

Let s_n denote the n-th partial summation of (1), so that

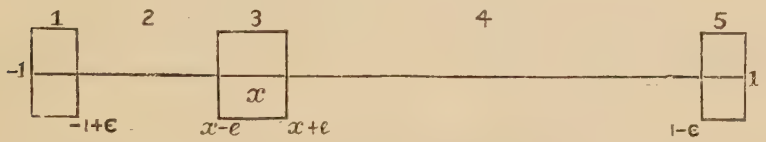
$$s_n(x) = \tfrac{1}{2}(n+1) \int_{-1}^{1} \{P_{n+1}(x)\, P_n(t) - P_n(x)\, P_{n+1}(t)\}\, f(t) \frac{dt}{x-t}. \tag{4}$$

We divide the interval of integration into five parts, and number these in order from left to right, denoting by a second index the corresponding part of the expression on the right for s_n, when we only integrate over the

partial interval in question. The partial intervals are as follows :—

(1) $(-1, -1+\epsilon)$; (2) $(-1+\epsilon, x-e)$; (3) $(x-e, x+e)$;

(4) $(x+e, 1-\epsilon)$; (5) $(1-\epsilon, 1)$;



where, firstly, e is chosen so small that this interval does not reach as far as the end-points -1 and 1, and secondly, ϵ is taken so small that the first and fifth intervals do not reach as far as the extremities of the third interval; *i.e.*

$$\epsilon < x+1-e, \quad \epsilon < 1-x-e. \tag{5}$$

18. In the second and fourth intervals $1/(x-t)$ and $\operatorname{cosec}\theta$, where $t = \cos\theta$, are both bounded functions, and therefore

$$\{f(t)\operatorname{cosec}\theta\}/(x-t)$$

is a summable function of t.

But using the well-known approximation,

$$P_n(t) = \sqrt{\left(\frac{2}{n\pi\sin\theta}\right)}\left[\cos\omega_n + O\left(\frac{1}{n}\right)\right]$$

where $$\omega_n = (n+\tfrac{1}{2})\theta - \frac{\pi}{4},$$

we have

$$\tfrac{1}{2}(n+1)\int_{-1+\epsilon}^{x-e} f(t)P_{n+1}(x)\,P_n(t)\,\frac{dt}{x-t}$$

$$= \frac{1}{\sqrt{(2\pi)}}\{\sqrt{(n+1)}\,P_{n+1}(x)\}\int_{-1+\epsilon}^{x-e} f(t)\sqrt{(\operatorname{cosec}\theta)}\left[\cos\omega_n + O\left(\frac{1}{n}\right)\right]\frac{dt}{x-t}.$$

Hence, using the theorem of Riemann-Lebesgue,

$$s_{n,2} \to 0 \quad (n \to \infty). \tag{6}$$

Similarly, $$s_{n\,4} \to 0 \quad (n \to \infty). \tag{6'}$$

19. We have now to consider the extreme intervals. Take the fifth $(1-\epsilon, 1)$.

We have, considering as before only one term of the integrand,

$$(n+1)\int_{1-\epsilon}^{1} f(t)\,P_{n+1}(x)\,P_n(t)\,f(t)\,\frac{dt}{x-t}$$

$$= \tfrac{1}{2}\sqrt{(n+1)}\,P_{n-1}(x) - \sqrt{(n+1)}\,\sqrt{n}\int_{1-\epsilon}^{1} f(t)\,P_n(t)\,\frac{dt}{x-t}$$

$$= O\int_{1-\epsilon}^{1}\frac{d\phi_n(t)}{x-t} \qquad (7)$$

where
$$\phi_n(t) \equiv \sqrt{n}\int_{1-\epsilon}^{t} f(t)\,P_n(t)\,dt. \qquad (8)$$

20. As a first step we shall now shew that, choosing η as small as we please,
$$|\phi_n(1)| < \eta, \qquad (9)$$
provided $\epsilon < \epsilon_\eta$, a quantity depending on η, and $n > n_\epsilon$, a quantity depending on ϵ.

To prove this, let first $f(t)$ be an *even* function and $n = 2n_1$, an *even* integer. Then $P_n(t)\,f(t)$ is an even function. Therefore, using (2) and (8),

$$a_n/\sqrt{n} = \left(\sqrt{n} + \frac{1}{2\sqrt{n}}\right)\int_{-1}^{1} f(t)\,P_n(t)\,dt \to 0 \quad (n\to\infty),$$

whence
$$\sqrt{n}\int_{-1}^{1} f(t)\,P_n(t)\,dt - 2\sqrt{n}\int_{0}^{1} f(t)\,P_n(t)\,dt \to 0. \qquad (10)$$

But, by the theorem of Riemann-Lebesgue,

$$\sqrt{n}\int_{0}^{1-\epsilon} f(t)\,P_n(t)\,dt = O\int_{0}^{1-\epsilon} f(t)\cos\omega_n t\,dt \to 0 \quad (n\to\infty).$$

Hence, by (10), when $f(t)$ is an even function,

$$\phi_n(1) = \sqrt{n}\int_{1-\epsilon}^{1} f(t)\,P_n(t)\,dt \to 0 \quad (n = 2n_1 \to \infty). \qquad (11)$$

Now, by a well-known formula,

$$P_{n+1}(t) = \tfrac{1}{2}\{(n+1)\,P_n(t) + (n+2)\,P_{n+2}(t)\} - (2n+3)\,t\,;$$

and therefore

$$\sqrt{(n+1)}\,P_{n+1}(t) = \tfrac{1}{2}\left(1+O\left(\frac{1}{n}\right)\right)\left[\sqrt{n}\,P_n(t) + \sqrt{(n+2)}\,P_{n+2}(t)\right] - t. \qquad (12)$$

Hence, in order to discuss $\phi_{n+1}(1)$, we merely have to discuss

$$\sqrt{n}\int_{1-\epsilon}^{1} f(t)\,P_n(t)\,\frac{dt}{t}.$$

But, by a theorem of Hobson's,*

$$|\phi_n(t)| \leqslant K\int_{1-\epsilon}^{t}(1-t^2)^{\frac{1}{4}}|f(t)|\,dt \leqslant K(1-t^2)^{\frac{1}{4}}\int_0^1 |f(t)|\,dt \leqslant L(1-t^2)^{\frac{1}{4}}, \tag{13}$$

using the First Theorem of the Mean, K and L being constants (independent of n, ϵ, and t).

Thus

$$\sqrt{n}\int_{1-\epsilon}^{1} f(t)\,P_n(t)\,\frac{dt}{t} = \int_{1-\epsilon}^{1}\frac{d\phi_n(t)}{t} = \left[\frac{\phi_n(t)}{t}\right]_{1-\epsilon}^{1} + \int_{1-\epsilon}^{1} t^{-2}\,\phi_n(t)\,dt. \tag{14}$$

Hence, by (13), $\left|\phi_n(1) - \sqrt{n}\int_{1-\epsilon}^{1} f(t)\,P_n(t)\,\frac{dt}{t}\right|$

is as small as we please, if ϵ is small enough.

Therefore, by (12),

$$\phi_{n+1}(1) \equiv \sqrt{(n+1)}\int_{1-\epsilon}^{1} f(t)\,P_{n+1}(t)\,dt = \left(1+O\,\frac{1}{n}\right)\tfrac{1}{2}\{\phi_n(1)+\phi_{n+2}(1)+2\eta\},$$

where, η being previously chosen as small as we please, ϵ is further restricted so that $\epsilon < \epsilon_\eta$, a quantity depending on η alone.

Hence, using (11), we have, n being now *even or odd*,

$$\operatorname*{Llt}_{n\to\infty} |\phi_n(1)| < \eta, \quad (\epsilon < \epsilon_\eta). \tag{15}$$

This has been proved for the case when $f(t)$ is an *even* function, but if $f(t)$ is an *odd* function we merely have to take n odd, and the proof of (11) applies. The rest of the argument applies as it stands for an odd or even function. Thus (15) has been proved for any odd or even function, and therefore for any function, since any function is the sum of an odd and an even function.

21. The remainder of the argument is precisely the same in character as that by which we deduced the case $n+1$ from the case n: the function $1/t$ is now replaced by $1/(x-t)$, which, by (5), is never infinite in the intervals we are considering.

---

* E. W. Hobson "On the Representation of a Function by a Series of Legendre's Functions", 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, p. 29.

Thus we find, for example, that, for $\epsilon < \epsilon_\eta$,

$$\underset{n\to\infty}{\text{Llt}} \left| \sqrt{n} \int_{1-\epsilon}^{1} f(t)\, P_n(t)\, \frac{dt}{x-t} \right| < \eta,$$

and hence

$$\underset{n\to\infty}{\text{Llt}} \mid s_{n,5} \mid < 2\eta, \quad (\epsilon < \epsilon_\eta). \tag{16}$$

Similarly,

$$\underset{n\to\infty}{\text{Llt}} \mid s_{n,1} \mid < 2\eta, \quad (\epsilon < \epsilon_\eta). \tag{16'}$$

Now, by (4), (6), (6'), (16), and (16'),

$$\underset{n\to\infty}{\text{Llt}} \mid s_n - s_{n,3} \mid < 4\eta, \quad (\epsilon < \epsilon_\eta). \tag{17}$$

But the left-hand side of (17) is independent of ϵ or η. Thus η being at our disposal, to choose as small as we like, the left-hand side of (17) can only be zero.

Hence

$$\underset{n\to\infty}{\text{Lt}} (s_n - s_{n,3}) = 0,$$

that is,

$$\underset{n\to\infty}{\text{Lt}} \left[s_n - \tfrac{1}{2}(n+1) \int_{x-e}^{x+e} \{ P_{n+1}(x)\, P_n(t) - P_n(x)\, P_{n+1}(t) \}\, f(t)\, \frac{dt}{x-t} \right] = 0, \tag{18}$$

e being *any* positive quantity, however small, less than $x+1$ and $1-x$.

22. The equation (18) proves the statement in the introduction as to the upper and lower functions. To prove that as to the peak and chasm functions, we have first to observe that, since the theorem of Riemann-Lebesgue, as Lebesgue brought out, involves uniform convergence, the convergence of $s_{n,2}$ and $s_{n,4}$ to their limit zero is uniform. Secondly, in (12), the term $O(1/n)$ does not involve t: thus in (15) the oscillation is uniform. The same is true in (16) and (16'). Thus the difference between s_n and the integral on the right in (18) converges *uniformly* to zero.

This proves the second statement in the enunciation.

ON SERIES OF BESSEL FUNCTIONS

By W. H. Young, Sc.D., F.R.S.

[Received December 1st, 1917.—Read December 6th, 1917.]

TABLE OF CONTENTS.

§§ 1. Introduction.
2. Definition of series of Bessel functions and notation.
3. Definition of Bessel series.
4. Definition and properties of Bessel functions.
5. The condition $A_r J_m(k_r z) \to 0$, necessary for convergence. Order of magnitude of $A_r J_m(k_r z)$ in the case of a Bessel series.

THEOREM.—*If* $x^{\frac{1}{2}+s} f(x)$ *is summable, where* $0 \leqslant s \leqslant m + \frac{1}{2}$, *then*

$$k^{-s} A_r J_m(k_r z) \to 0 \quad (r \to \infty).$$

6. Connection of the preceding theorem with Cesàro summation.

THEOREM.—*The convergence* (Ck) *of a restricted Fourier series*

$$\Sigma (a_n \cos nx + b_n \sin nx),$$

at any point internal to $(-\pi, \pi)$, *depends only on the nature of the associated function in the neighbourhood of the point, provided*

$$n^{-k} (a_n \cos nx + b_n \sin nx) \to 0 \quad (n \to \infty).$$

7. Lemma from the theory of repeated integration.
8. Proof by the complex variable of the expansion of x^m in a uniformly convergent Bessel series $x^m = (m + H) \Sigma C_r k_r^{-2} J_m(k) J_m(k_r x)$.
9. THEOREM.—*If the coefficients of the Bessel series of a function are all zero, the function is a null-function.*
10. THEOREM.—*If two functions have the same Bessel series, they only differ by a null-function.*

COR.—*If a series of Bessel functions converge uniformly, it is the Bessel series of a continuous function, and this continuous function is determined uniquely and is the sum-function.*

11–13. THEOREM.—*If* $\Sigma A_r J_m(k_r z)$ *is the Bessel series of* $f(z)$, *and satisfies the condition* $A_r J_m(k_r z) \to 0$ $(r \to \infty)$, *the series*

$$\Sigma V_r = \Sigma A_r \int_z^1 z^{-m+1} \left\{ \int_0^z t^{\frac{1}{2}m} J_m\{k_z \sqrt{t}\}\, dt \right\} dz$$

converges uniformly to the function

$$\psi(z) = \int_z^1 z^{-m-1} \left\{ \int_0^z t^{\frac{1}{2}m} f(\sqrt{t})\, dt \right\} dz,$$

and, when multiplied by $z^{\frac{1}{2}m}$, *is the uniformly convergent Bessel series of the function* $z^{\frac{1}{2}m}\psi(z)$ *in* $(0 < z < 1)$.

14. THEOREM.—*The second integrated series of the Bessel series* $\Sigma A_r J_m(k_r z)$ *of* $f(z)$ *converges uniformly in* $(0 < z < 1)$ *to* $\int dz \int f(z)\,dz$.

15–31. THE FUNDAMENTAL THEOREM.—*The Bessel series* $\Sigma A_r J_m(k_r z)$ *of* $f(z)$ *behaves in respect of convergence, divergence or oscillation, uniform or otherwise, precisely like the Fourier series of* $f(z)$, *at any point internal to* $0 < z < 1$, *provided*

$$A_r J_m(k_r z) \to 0 \quad (r \to \infty).$$

15–24. *Proof in the special case when m is rational.*

25, 26. *General case when m is irrational, and reduction of the problem to the discussion of a non-harmonic trigonometrical series, whose terms tend to zero, and whose second integrated series converges to the integral of an integral.*

27–30. Digression on the theory of non-harmonic trigonometrical series.

THEOREM.—*The upper and lower functions of a non-harmonic trigonometrical series* $\Sigma\{a_r \cos(r+k)x + b_r \sin(r+k)x\}$, *where* $a_r \to 0$, $b_r \to 0$, *and the sum of the second integrated series is the integral of an integral of* $f(x)$ *in the interval considered, are the same as those of the Fourier series of* $f(x)$, *and the difference of corresponding partial summations tends uniformly to zero.*

31. Conclusion of the proof of the fundamental theorem of § 15.

32. COR. 1.—*The first integrated series of a Bessel series whose terms tend to zero converges uniformly to an integral in* $(0 < z < 1)$.

33. COR. 2.—*The fundamental theorem remains true for a series of Bessel functions* $\Sigma A_r J_m(k_r z)$, *whose terms tend to zero, provided in some interval containing the point, or points, in question the sum of the second integrated series is the integral of an integral.*

Hence the first integrated series of such a series converges to an integral.

34. THEOREM.—*If* $f(x)$ *is, in a neighbourhood surrounding the point x, the integral of the integral of* $f(x)$, *a function whose Fourier series converges at that point, then the derived series of the Bessel series of* $f(x)$ *will converge at the point x to* $f'(x)$.

35. THEOREM.—*If* (i) $x^{m+\frac{1}{2}} \dfrac{d}{dx}\{x^{-m} f(x)\}$ *is summable in* $(0, 1)$, *and* (ii) $x^{m+1} f(x) \to 0$ $(x \to 0)$, *then* $k_r^{\frac{3}{2}} \int_0^1 J_m(k_r x)\, x f(x)\,dx \to 0 \quad (r \to \infty)$.

36–38. Appendix on the convergence of the series $\Sigma A_r J_m(k_r x)$ at the point $x = 1$.

36. Three auxiliary theorems. Proof of the first. Proof of the third in § 8.

37. Proof of the second.

38. THEOREM.—*If* $f(x)$ *has bounded variation at* $x = 1$, *its Bessel series converges there to* $f(1-0)$.

---

1. The theory of these series as hitherto exposed may be regarded as involving serious difficulties of analysis. Till quite recently Dini's work constituted the last word on the subject. Previously a long series of mathematicians had occupied themselves with the question, including names like Fourier, Hankel, and Harnack. Yet Dini seems to have been

the first to obtain reliable results. The conditions due to him are somewhat complicated in character, and, in the light of recent advances in Analysis, more especially in the theory of Fourier series, far from final. Kneser has availed himself of the modern subject of Integral Equations to obtain results of an isolated nature for Bessel series $\Sigma A_r J_m(k_r x)$, where

$$A_r = C_r \int_0^1 J_m(k_r x)\, x f(x)\, dx,$$

for the case $m = 0$, and Hobson, basing himself, as he tells us, partly on Kneser's work, has proved that the Bessel series of a function $f(x)$ converges at every point of the half-closed interval $(0 < x < 1)$ at which the function has bounded variation, provided only that $\sqrt{x}\, f(x)$ is summable.

I have already, in a paper dealing with series of Legendre's Polynomials, explained a general method of treating the expansion of $f(x)$ in a series of functions, based on a fundamental theorem which I have recently discovered in the theory of what I have called Restricted Fourier Series. The application to Legendre's polynomials exhibits the method in a peculiarly pure light, as the asymptotic approximation for $P_n(x)$ involves trigonometrical functions of the type employed in harmonic trigonometrical series, viz. $\sin nx$ and $\cos nx$, where n is an integer. Moreover the fact that the functions in terms of which we expand are polynomials enables us to avoid entirely the use of the complex variable.

In order then to exhibit the full power of the method, it has seemed to me desirable to discuss a case where these simplifications do not present themselves. I have chosen the theory of series of Bessel functions $\Sigma A_r J_m(k_r x)$ with this object in view, and the more readily because, as has been just pointed out, so little has hitherto been achieved in the theory of these series.

I have been at pains to explain the processes employed as carefully as possible, so as to render them intelligible to those who are unacquainted with previous work on the subject. There is indeed little connexion between the work here exposed and that of previous authors.

As regards the conclusions arrived at, and indicated, these may, I think, fairly be said to be of a far-reaching nature, and constitute little less than the complete solution of the problem of determining the behaviour of the series under consideration in the matter of convergence, uniform convergence, convergence by the methods of Cesàro, oscillation, and so forth, at any *internal* point of the interval (0, 1). Thus it is shown that, *under the sole condition—necessary for convergence—that the typical term of a Bessel series tends to zero, the series behaves at any such internal*

point precisely like a Fourier series, and we may obtain, without difficulty, circumstances under which it will converge in the Cesàro manner, from a knowledge of the power to which we must raise the inverse of the index expressing position in the series in order that the typical term, multiplied by this power, should tend to zero. On the other hand very general sufficient conditions, which the function $f(x)$ must satisfy in order that the coefficients may be of the required order, can be at once written down.

One of the more interesting features of the discussion is perhaps the introduction of trigonometrical series of the non-harmonic type

$$\Sigma \{a_n \cos(k+n)x + b_n \sin(k+n)x\},$$

which are shown to have properties analogous to those possessed by ordinary trigonometrical series, equally whether k be rational or irrational.

Again, it is of importance to remark that the scope of the investigation does not only include series of the type hitherto studied (*Bessel series*) in which the coefficients are expressible by means of integrals involving the function $f(x)$. Our method gives results of scarcely less importance when the coefficients are not of this form (*series of Bessel functions*), in accordance with the analogy suggested by the step taken in passing from ordinary Fourier series to Restricted Fourier Series. In particular we are able to deal with the derived series of Bessel series, although such derived series in our case necessarily cease to have the same form as the series from which they are derived. We are thus able to complete in a notable point the usual investigations in the applications. In fact, *it has been usual to assume tacitly that, if the individual term satisfies a certain differential equation, the function represented by the sum of an infinite series of such terms will also satisfy it. It is here shewn that such a conclusion is legitimate, if the function $f(x)$ have, in the interval in which such an equation is to be satisfied, a differential coefficient of suitable order, such that the Fourier series of a function equal to this differential coefficient in this interval, and having any convenient values elsewhere, converges in the interval in question. This is, for instance, the case if the differential coefficient has bounded variation.*

It* will be seen that our method, based as it is on the asymptotic trigonometric approximation [(viii) of § 4], which only holds in an interval excluding the origin, gives us no information as to what happens at $z = 0$. When $m > 0$, the series obviously converges there to zero, but

* Added January 29th, 1918.

it remains an open question as to the conditions under which the convergence is bounded in the neighbourhood, or, in the case when the function approaches zero as limit there, uniform. On the other hand we are able, by considerations analogous to those employed below, to deal with the end-point $z = 1$, obtaining a result exactly corresponding to that for an internal point of the interval (0, 1). The exposition of these latter considerations involves, however, a more detailed treatment of the theory of non-harmonic trigonometrical series than has been found possible to include in the present communication.* The question, moreover, of convergence of a series, at the extremity of an interval in which it defines an associated function, requires and deserves a fuller and separate discussion, and I propose shortly to make a communication to the Society on these and kindred matters.

On the other hand, in view of the fact that I have not found it convenient to dispense entirely with the use of the complex variable, I have thought it well to shew in an Appendix what additional considerations, drawn from the theory of the complex variable alone, and avoiding, in particular, the use of the theory of Integral Equations, suffice for the proof of the known theorem that a Bessel series converges at $z = 1$, when its associated function $f(z)$ has bounded variation there and $\sqrt{z}\, f(z)$ is summable.

It should finally be remarked that the results of the paper hold as they stand, not only for all values of $m \geqslant 0$, but also for $m \geqslant -\frac{1}{2}$, and even for all values of m greater than -1, provided slight modifications be made, involving a change in the condition of summability in the neighbourhood of the origin of the function $f(z)$ associated with the series.

2. The series which we are going to investigate are *series of Bessel functions*

$$\Sigma A_r J_m(k_r z), \tag{1}$$

where m is a fixed positive quantity (or zero), and the numbers $k_1, k_2, \ldots, k_r, \ldots$ form a monotone increasing succession tending to infinity, and are deter-

* From the form of the first approximation given in (25a) of § 25, these conditions of convergence can be deduced. We have, in fact, a series of cosines of $(k + r)\, z$, where $\pi z = \pi - y$, so that there is symmetry about the origin of y. It then follows from a general theorem, given by me in a recent communication to the Royal Society, that this non-harmonic trigonometrical series behaves in respect of convergence and uniform convergence at $y = 0$, exactly like the expansion in such a series of a function which is summable in the interval $(0, 2\pi)$, and, at and in the neighbourhood of the origin $y = 0$, is equal to $f(1 - y/\pi)$. By the theory of such expansions we then perceive the truth of the statement made in the text.

mined as the positive roots of the transcendental equation

$$kJ'_m(k)+HJ_m(k)=0, \tag{2}$$

involving a parameter H at our disposal.* We may accordingly write

$$k_r\pi=\tfrac{1}{2}m+\tfrac{1}{4}+s+r+r^{-1}a(r), \tag{3}$$

where s is a certain fixed integer, and $a(r)$ is a bounded function of r of the form

$$a(r)=C+r^{-1}a_1(r),$$

$a_1(r)$ being another bounded function of r.

3. The series (1) is said to be *a Bessel series*, if the coefficients A_r have the form

$$A_r=C_r\int_0^1 J_m(k_rt)\,tf(t)\,dt=\tfrac{1}{2}C_r\int_0^1 f\sqrt{t}\,J_m(k_r\sqrt{t})\,dt, \tag{4}$$

where C_r is given by the equations

$$\tfrac{1}{2}C_r=1\Big/\int_0^1\{J_m(k_r\sqrt{t})\}^2\,dt=1\Big/\left\{\left(1-\frac{m^2}{k_r^2}\right)J_m^2(k_r)+J_m'^2(k_r)\right\}$$

$$=k_r^2/\{H^2+k_r^2-m^2\}\,J_m^2(k_r)=O(k_r). \tag{5}$$

4. I recall the definition of Bessel's function $J_m(x)$,†

$$J_m(x)=\frac{x^m}{2^m\,\Gamma(m+1)}\left\{1-\frac{x^2}{2(2m+2)}+\frac{x^4}{2.4(2m+2)(2m+4)}+\dots\right\},$$

this being valid for all values of x other than real and negative values. The following known properties of $J_m(x)$ are required in the sequel:—

(i) $J_m(x)$ is a bounded function of x in the infinite interval $0\leqslant x$; indeed $\sqrt{x}\,J_m(x)$ is a bounded function of x for $0\leqslant x$;‡ $x^{-m}J_m(x)$ is evidently bounded in any finite interval $(0, B)$;

---

* E. W. Hobson, "On the Representation of a Function by a Series of Bessel's Functions", 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 359–388. In the formula on p. 366,

$$\lambda_r^{\frac{1}{2}}=k_r.$$

† I. Todhunter, *An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions* (1875), Chapters XXX *et seq.*

‡ Hobson, *loc. cit.*, p. 366.

(ii) the differential equation satisfied by $J_m(x)$ is

$$x\frac{d}{dx}[xJ'_m(x)]-(m^2-x^2)J_m(x)\equiv J''_m+x^{-1}J'_m+(1-m^2x^{-2})J_m=0;$$

(iii) $$mJ_m(x)=\tfrac{1}{2}x\{J_{m-1}(x)+J_{m+1}(x)\};$$

(iv) $$J'_m(x)=\tfrac{1}{2}\{J_{m-1}(x)-J_{m+1}(x)\};$$

(v) $$\int_0^1 J_m(k_r\sqrt{t})\,J_m(k_n\sqrt{t})\,dt=0 \quad (n\neq r)$$

[the value of this integral when $n=r$ is $2/C_r$, as in (5) above];

(vi) $$\int_0^x t^{m+1}J_m(t)\,dt=x^{m+1}J_{m+1}(x);$$

(vii) $$\int_0^x t^{-\frac{1}{2}(m+1)}J_{m+1}(\sqrt{t})\,dt=2\left\{\frac{1}{2^m m}-x^{-\frac{1}{2}m}J_m(\sqrt{x})\right\};$$

(viii) $$\sqrt{(\tfrac{1}{2}\pi x)}\,J_m(x)$$
$$=\cos\psi+\frac{m^2-\frac{1}{4}}{2x}\sin\psi-\frac{(m^2-\frac{1}{4})(m^2-\frac{9}{4})}{2!\,(2x)^2}\left[\cos\psi+\theta\frac{m^2-\frac{16}{4}}{3(2x)}\right],*$$

where $|\theta|\leqslant 1$, and $\psi=-x+\frac{1}{4}(2m+1)\pi$.

From this *asymptotic expression* for $J_m(x)$, we at once get the following, using (iv), and remarking that the change of m into $m-1$ subtracts $\frac{1}{2}\pi$ from ψ, and that into $m+1$ adds $\frac{1}{2}\pi$:—

(ix) $$\frac{d}{dx}\{\sqrt{(\tfrac{1}{2}\pi x)}\,J_m(x)\}$$
$$=\sqrt{(\tfrac{1}{2}\pi x)}\{J'_m(x)+J_m(x)/2x\}$$
$$=\sin\psi-\frac{n^2-\frac{1}{4}}{2x}\cos\psi-\frac{(n^2-\frac{1}{4})(n^2+\frac{7}{4})}{2!\,(2x)^2}\sin\psi+O(x^{-3}).$$

5. In the first instance we are especially interested in those series for which the typical term tends to zero, *i.e.*

$$A_r J_m(k_r z)\to 0 \quad (r\to\infty). \tag{6}$$

This condition is evidently *necessary*, if the series (1) is to converge, and it will be seen that its assumption renders the series as manageable as a

* Lipschitz, *Journal für Math.*, Vol. 56, 1859, pp. 189–196: reproduced in A. Gray and G. B. Mathews, *A Treatise on Bessel Functions*, 1895, p. 70.

Fourier series. It is not satisfied of itself when the series is a Bessel series. In fact the order of magnitude of the typical term varies with the properties of the associated function $f(x)$.

It is immediately evident that no more is required, in order that the usual expression for the coefficients A_r, viz. (4), should have a meaning, than that $x^{m+1}f(x)$ should be absolutely integrable (summable). On the other hand, it is easily seen that the absolute integrability of $\sqrt{x}f(x)$ is sufficient to ensure the convergence of the typical term to zero.

More generally we have the following theorem:—

THEOREM.—*If* $x^{\frac{1}{2}+s}f(x)$ *is absolutely integrable, where* $0 \leqslant s \leqslant m+\frac{1}{2}$, *then*

$$r^{-s}A_r J_m(k_r z) \to 0 \quad (r \to \infty).$$

Case 1.—Let $0 < s \leqslant \frac{1}{2}$.

Then, if $0 < z \leqslant 1$, $z^{\frac{1}{2}-s} \leqslant 1$,

and if $1 < z$, $z^{\frac{1}{2}-s} < z^{\frac{1}{2}}$.

Thus in either case $z^{\frac{1}{2}-s}J_m(z)$ is bounded, since $J_m(z)$ and $\sqrt{z}J_m(z)$ are both bounded.

Case 2.—Let $\frac{1}{2} < s \leqslant m+\frac{1}{2}$.

Then, if $0 < z \leqslant 1$, $z^{\frac{1}{2}-s} < z^{-m}$,

and if $1 < z$, $z^{\frac{1}{2}-s} < 1$.

Thus in either case $z^{\frac{1}{2}-s}J_m(z)$ is bounded, since $z^{-m}J_m(z)$ and $J_m(z)$ are both bounded.

Thus in all cases $z^{\frac{1}{2}-s}J_m(z)$ is bounded.

Now
$$A_r/C_r = \int_0^1 J_m(k_r t)\, t\, f(t)\, dt$$
$$= k_r^{s-\frac{1}{2}} \int_0^e \{(k_r t)^{\frac{1}{2}-s} J_m(k_r t)\}\, t^{\frac{1}{2}+s} f(t)\, dt$$
$$+ k^{-\frac{1}{2}} \int_e^1 \{(k_r t)^{\frac{1}{2}} J_m(k_r t)\}\, t^{\frac{1}{2}} f(t)\, dt.$$

Using in the second integral the asymptotic formula (viii), we see that it vanishes, by the theorem of Riemann-Lebesgue, when $r \to \infty$. Denoting by B the upper bound of $|(k_r t)^{\frac{1}{2}-s} J_m(k_r t)|$, the modulus of the first integral does not exceed

$$B \int_0^e |t^{\frac{1}{2}+s} f(t)|\, dt,$$

which is as small as we please, if e be chosen small enough. Thus this term determines the order $o(k_r^{s-\frac{1}{2}})$ of A_r/C_r.

Hence, since $C_r = O(k_r)$, and $k_r^{\frac{1}{2}} J_m(k_r z)$ is bounded,

$$k_r^{-s} A_r J_m(k_r z) \to 0.$$

This proves the theorem, since k_r is of order r.

6. The preceding theorem is important in connexion with the Cesàro summation of Bessel series, in virtue of the following theorem respecting the Cesàro convergence of Restricted Fourier Series :—

Theorem.—*The convergence (Ck) of a Restricted Fourier Series at any point, internal to the interval* $(-\pi < x < \pi)$, *depends only on the nature of the associated function in the neighbourhood of the point, provided only the typical term tends to zero when divided by the k-th power of the integer expressing its place in the series.**

For the proof of this theorem I refer to a forthcoming paper to be published elsewhere, as I do not propose to extend the length of the present communication by entering into a discussion of Cesàro convergence. It seems, however, relevant to the matter in hand to state and prove the above theorem respecting the convergence to zero of the typical Bessel term.

7. We proceed now to the discussion of the convergence of a Bessel series satisfying (6).

We shall require the following simple lemma :—

Lemma.—*If* dg/dt *and* dh/dt *exist and are bounded functions of* t, *and*

---

* The importance of this result in the theory of the Cesàro convergence of series in general cannot be overrated. Thus, in the case of Legendre series, it at once leads to the theorem that *the Legendre series of* $f(x)$ *converges* (Ck), *where* $k \geqslant \frac{1}{2}$, *whenever the Fourier series of* $f(\cos\theta)$, *where* $x = \cos\theta$, *converges* (Ck), *where* $k > 0$.

In the case of Bessel series, a similar statement is true, as follows from the argument in the text, whenever $xf(x)$ is absolutely integrable. More generally, *if* $x^s f(x)$ *is absolutely integrable, the Bessel series of* $f(x)$ *converges* $(C, s+\frac{1}{2})$, *where* $0 \leqslant s < \frac{1}{2}$, *whenever the Fourier series of* $f(x)$ *converges* (Ck), *where* $k > 0$. The corresponding statements when $s \geqslant \frac{1}{2}$ may be left to the reader who has mastered the various papers concerned.

The difference between the Legendre and Bessel series will be easily understood; the convergence of the typical Legendre term to zero is secured by $(1-x^2)^{-\frac{1}{4}} f(x)$ being absolutely integrable, while, if this condition be not imposed, the typical Legendre term only tends to zero when divided by $\sqrt{n}$.

the repeated integral series

$$\Sigma \int g(t)\,dt \int h(t)\,u_r(t)\,dt$$

*converges uniformly to a repeated integral**

$$\Phi(t) \equiv \int g(t)\,dt \int h(t) f(t)\,dt,$$

then the second integrated series of $\Sigma u_r(t)$ *converges uniformly to the repeated integral of* $f(t)$, *i.e.*

$$\Sigma \int dt \int u_r(t)\,dt = \int dt \int f(t)\,dt.$$

The converse is also true.

Let us write

$$s_r = \sum_1^r \int u_r(t)\,dt, \qquad \sigma_r = \sum_1^r \int h(t)\,u_r(t)\,dt, \qquad \eta(t) = 1/h(t),$$

$$S_r = \sum_1^r \int dt \int u_r(t)\,dt, \quad \Sigma_r = \sum_1^r \int g(t)\,dt \int h(t)\,u_r(t)\,dt, \quad \gamma(t) = 1/g(t).$$

Then
$$g(t)\,\sigma_r(t) = \frac{d}{dt}\,\Sigma_r(t)$$

and therefore

$$s_r = \int \eta(t)\,\frac{d}{dt}\,\sigma_r(t)\,dt = \eta(t)\,\sigma_r(t) - \int \eta'(t)\,\gamma(t)\,\frac{d}{dt}\,\Sigma_r(t)\,dt$$

$$= \eta(t)\,\sigma_r(t) - \eta'(t)\,\gamma(t)\,\Sigma_r(t) + \int \frac{d}{dt}\,\{\eta'(t)\,\gamma(t)\}\,\Sigma_r(t)\,dt.$$

Hence, integrating,

$$S_r = \eta(t)\,\gamma(t)\,\Sigma_r(t) - \int \frac{d}{dt}\,\{\eta(t)\,\gamma(t)\}\,\Sigma_r(t)\,dt - \int \eta'(t)\,\gamma(t)\,\Sigma_r(t)\,dt$$

$$+ \int dt \int \frac{d}{dt}\,\{\eta'(t)\,\gamma(t)\}\,\Sigma_r(t)\,dt.$$

* Which can, of course, be expressed in the form given to $\Phi(t)$ since, $g'(t)$ and $h'(t)$ being bounded, $g(t)$ and $h(t)$ are integrals, and the hypothesis might be written

$$\Sigma \int dG \int u_r(t)\,dH = \int dG \int f(t)\,dH,$$

where G and H are the integrals of g and h.

Similarly,

$$\int dt \int f(t)\, dt = \eta(t)\, \gamma(t)\, \Phi(t) - \int \frac{d}{dt} \{\eta(t)\, \gamma(t)\} \Phi(t)\, dt - \int \eta(t)\, \gamma(t)\, \Phi(t)\, dt$$
$$+ \int dt \int \frac{d}{dt} \{\eta'(t)\, \gamma(t)\} \Phi(t)\, dt.$$

But since the series whose partial summation is Σ_r converges uniformly to $\Phi(t)$, the right-hand side of the last equation is the sum of the series whose partial summation is the right-hand side of the preceding one. Hence the required result at once follows.

The converse only requires the change of g into γ, and h into η, &c.

8. We next require the correlative in our theory of Lerch's theorem, that is, the theorem that *if two functions have the same Bessel series they only differ by a null-function, and, if continuous, are identical.*

For this purpose we need a subsidiary result, the proof of which is best obtained by using the complex variable.

Let us write [using $f(t)$ and $F(t)$ now without any reference to our former use of the symbol $f(t)$, but merely in order to maintain the notation in Lindelöf, *Calcul des Résidus*, pp. 20, 21]

$$f(t) = \frac{tJ'_m(t)}{J_m(t)} + H,$$

and therefore, using (iv),

$$f'(t) = \frac{(m^2 - t^2)\, J_m^2 - t^2 J_m'^2}{tJ_m^2}.$$

The zeros of $f(t)$ are then, by (2), our values $k_1, k_2, \ldots, k_r, \ldots$, taken both positively and negatively, none being zero; and its poles are the zeros of $J_m(t)$, excluding the origin, these necessarily distinct from the former.

Now put

$$F(t) = -\frac{J_m(tx)}{tJ_m(t)\, f'(t)} = \frac{J_m(tx)\, J_m(t)}{t^2} \Big/ \frac{1}{t^2} \{(t^2 - m^2)\, J_m^2 + t^2 J_m'^2\},$$

and integrate round a contour C, excluding those, say r, of the poles of $f(t)$ that lie within the contour by small circles K, the origin by a small circle O, and the roots of

$$(m^2 - t^2)\, J_m^2 - t^2 J_m'^2 = 0,$$

none of which coincide with k's, by small circles L. Then we get, by the

usual rule,

$$\frac{1}{2\pi i}\int_{C+K+O+L}\frac{F(t)f'(t)}{f(t)}\,dt = 2\{F(k_1)+F(k_2)+\ldots+F(k_r)\}.$$

The integrand here may be written

$$-\frac{J_m(tx)}{tJ'_m(t)+HJ_m}.$$

As the excluded poles of $f(t)$ are not poles of this function, we may now erase the corresponding circles K, and also the circles L, for the same reason, as $f(t)$ and $f'(t)$ do not vanish together.

Hence we have

$$-\frac{1}{2\pi i}\int_C \frac{J_m(tx)\,dt}{t\{HJ_m(t)+tJ'_m(t)\}}+\frac{1}{2\pi i}\int_O \frac{J_m(tx)\,dt}{t\{HJ_m(t)+tJ'_m(t)\}} = 2\sum_{n=1}^{r} F(k_n).$$

The second integral gives us then a residue

$$-\operatorname*{Lt}_{t=0}\frac{J_m(tx)}{HJ_m(t)+tJ'_m(t)} = -\frac{x^m}{H+m}.$$

Now the value of the first integral tends to zero when the contour C tends to infinity, as will be found if we use the known asymptotic expressions for J_m and J'_m* for complex values of the variable. The integrand then becomes of the order $1/(\sqrt{x}\,t^2)$. Hence, finally,

$$2\sum_{r=1}^{\infty} F(k_r) = -\frac{x^m}{m+H},$$

which is the result required and may be written in the form

$$x^m = (m+H)\,\Sigma C_r k_r^{-2} J_m(k_r)\,J_m(k_r x), \tag{7}$$

C_r being given by the equation (5).

9. We can now prove the following theorem, from which our extension of Lerch's theorem evidently follows immediately:—

THEOREM.—*If the coefficients of the Bessel series of a function $g(x)$ are all zero, $g(x)$ is a null-function.*

For we have proved (§ 8) by the complex variable that

$$\frac{x^m}{m+H} = \Sigma k_r^{-2}\,C_r J_m(k_r x)\,J_m(k_r). \tag{7}$$

---

* For instance, formula (4) in Hobson's paper, *loc. cit.*, p. 364.

This series evidently converges uniformly in the closed interval $0 \leqslant x \leqslant B$. Hence, putting $x = \sqrt{z}$, we may multiply both sides by $z^{\frac{1}{2}m}$ and integrate term-by-term, from 0 to z. We thus get, by (vi),

$$z^{m+1}/(m+H)(m+1) = 2z^{\frac{1}{2}(m+1)} \Sigma k_r^{-3} C_r J_{m+1}(k_r\sqrt{z})\, J_m(k_r),$$

that is, $$z^{\frac{1}{2}(m+1)}/(m+H)(m+1) = 2\Sigma k_r^{-3} C_r J_{m+1}(k_r\sqrt{z})\, J_m(k_r).$$

Now multiply by $z^{-\frac{1}{2}(m+1)}$ and integrate between 0 and z. We get, by (vi),

$$\frac{z}{(m+H)(m+1)} = 4\Sigma \left\{ \frac{1}{2^m m!} - x^{-\frac{1}{2}m} J_m(k_r\sqrt{z}) \right\} J_m(k_r)\, C_r k_r^{-4},$$

or
$$\frac{z^{\frac{1}{2}m+1}}{(m+H)(m+1)} = 4\Sigma C_r k_r^{-4} J_m(k_r)\, z^{\frac{1}{2}m}/2^m m! - 4\Sigma C_r k_r^{-4} J_m(k_r)\, J_m(k_r\sqrt{z}). \quad (8)$$

This process may be repeated any number of times and we get similar equations for $z^{\frac{1}{2}m+p}$ for all integral values of p. From (7) we have, if $g(\sqrt{z})$ is an absolutely integrable function,

$$\int z^{\frac{1}{2}m}\, g(\sqrt{z})\, dz = (m+H)\, \Sigma C_r k_r^{-2} J_m(k_r) \int_0^1 J_m(k_r\sqrt{z})\, g(\sqrt{z})\, dz = 0,$$

under the supposition that the coefficients of the Bessel series of $g(\sqrt{z})$ are all zero.

Hence, also from (8), in the same circumstances,

$$\int_0^1 z^{\frac{1}{2}m+1}\, g(\sqrt{z})\, dz = 0,$$

and, from the similar equations with any integer p in place of 1,

$$\int_0^1 z^{\frac{1}{2}m+p}\, g(\sqrt{z})\, dz = 0.$$

Hence, by the extended theorem of Lerch, $z^{\frac{1}{2}m}\, g(\sqrt{z})$ is a null function, and therefore so is also $g(\sqrt{z})$. Moreover, if $g(\sqrt{z})$ is continuous in the open interval $(0 < x)$ it is necessarily zero for $0 < x$.

This proves the theorem.

10. Now if $f_1(t)$ and $f_2(t)$ have the same Bessel series, and we put

$$g(\sqrt{t}) = f_1(t) - f_2(t),$$

it follows, from the formal expression (4) for the coefficients of a Bessel series, that the coefficients of the Bessel series of $g(\sqrt{t})$ are all zero.

Hence by the preceding theorem $g(\sqrt{t})$ is a null function. Thus :—

THEOREM (*Lerch's Theorem for Bessel functions*).—*If two functions have the same Bessel series they only differ by a null-function, and, if they are continuous, they are identical.*

COR.—*If a series of Bessel functions converges uniformly, it is the Bessel series of a continuous function, and this continuous function is determined uniquely, and is the sum-function.*

For let
$$g(\sqrt{t}) = \Sigma A_r J_m(k_r \sqrt{t}), \tag{1}$$
where the convergence is uniform.

Let us multiply (1) term-by-term by $\frac{1}{2}C_r J_m(k_r \sqrt{t})$, and integrate from 0 to 1. Since (1) converges uniformly, this is allowable. Using the formulæ (5) and (iii), we then have, as the result of integration,

$$\tfrac{1}{2}C_r \int_0^1 g(\sqrt{t})\, J_m(k_r \sqrt{t})\, dt = A_r,$$

for all values of r. This shews that (1) is the Bessel series of $g(\sqrt{t})$.

But $g(\sqrt{t})$ is the sum of a uniformly convergent series of continuous functions, and is therefore continuous. Thus, by the preceding theorem, if the series (1) is also the Bessel series of $f(t)$ we see, comparing the two expressions obtained for C_r, that $f(\sqrt{t})$ and $g(\sqrt{t})$ can only differ by a null-function, and that, if $f(\sqrt{t})$ is continuous, it is identical with $g(\sqrt{t})$.

This proves the corollary.

11. We have next to prove that the second integrated series converges uniformly to a repeated integral. We deduce this from the following theorem :—

THEOREM.—*If the series* $\Sigma A_r J_m(k_r z)$, (1)

satisfying the condition $k_r^{-\frac{1}{2}} A_r \to 0 \quad (r \to \infty)$,

is the Bessel series of a function $f(z)$, *the series*

$$\Sigma V_r = \Sigma A_r \int_z^1 z^{-(m+1)} \left\{ \int_0^z t^{\frac{1}{2}m} J_m(k_r \sqrt{t})\, dt \right\} dz$$

converges uniformly to a function $\psi(z)$ *which is the integral of an integral, and has the form*

$$\psi(z) = \int_z^1 z^{-(m+1)} \left\{ \int_0^z t^{\frac{1}{2}m} f(\sqrt{t})\, dt \right\} dz.$$

By the formula (vi) we have at once

$$\int_0^z t^{\frac{1}{2}m} J_m\{k_r\sqrt{t}\}\,dt = \frac{2}{k_r} z^{\frac{1}{2}(m+1)} J_{m+1}(k_r\sqrt{z}). \tag{9}$$

Therefore, multiplying both sides by z^{-m-1}, and integrating

$$\begin{aligned}\int_z^1 z^{-m-1}dz \int_0^z t^{\frac{1}{2}m} J_m(k_r\sqrt{t})\,dt &= \frac{2}{k_r}\int_z^1 z^{-\frac{1}{2}(m+1)} J_{m+1}(k_r\sqrt{z})\,dz \\ &= \frac{2}{k_r}\int_{k_r^2 z}^{k_r^2} k_r^{m-1} u^{\frac{1}{2}(m+1)} J_{m+1}(\sqrt{u})\,du \\ &= -\frac{4}{k_r}\int_{k_r^2 z}^{k_r^2} k_r^{m-1}\frac{d}{du}\{u^{-\frac{1}{2}m} J_m(\sqrt{u})\}\,du \\ &= \frac{4}{k_r^2} z^{-\frac{1}{2}m} J_m(k_r\sqrt{z}) - \frac{4}{k_r^2} J_m(k_r). \end{aligned} \tag{10}$$

We notice particularly here that the second term of the right-hand side is got from the first by putting $z = 1$.

Now J_m is a bounded function, and by hypothesis $k_r^{-\frac{1}{2}} A_r \to 0$, and is therefore bounded. Thus the result just obtained shews that *the series* ΣV_r *is a uniformly convergent series.*

12. Now
$$A_r = \tfrac{1}{2} C_r \int_0^1 f(\sqrt{t})\, J_m(k_r\sqrt{t})\,dt. \tag{4}$$

Let us write
$$\phi(t) = \int_0^t u^{\frac{1}{2}m} f(\sqrt{u})\,du,$$

$$\psi(t) = \int_z^1 \phi(u)\, u^{-(m+1)}\,du;$$

then

$$\begin{aligned}\int_0^1 f(\sqrt{t})\, J_m(k_r\sqrt{t})\,dt &= \int_0^1 \phi'(t)\, t^{-\frac{1}{2}m} J_m(k_r\sqrt{t})\,dt \\ &= \Big[\phi(t)\, t^{-\frac{1}{2}m} J_m(k_r\sqrt{t})\Big]_0^1 + \tfrac{1}{2}\int_0^1 k_r\phi(t)\, t^{-\frac{1}{2}(m+1)} J_{m+1}(k_r\sqrt{t})\,dt \\ &= \phi(1)\, J_m(k_r) - \tfrac{1}{2}\int_0^1 k_r\psi'(t)\, t^{\frac{1}{2}(m+1)} J_{m+1}(k_r\sqrt{t})\,dt \\ &= \phi(1)\, J_m(k_r) + \tfrac{1}{4}\int_0^1 k_r^2\psi(t)\, t^{\frac{1}{2}m} J_m(k_r\sqrt{t})\,dt. \end{aligned}$$

Thus
$$A_r = \tfrac{1}{2} C_r \Big\{\phi(1)\, J_m(k_r) + \tfrac{1}{4}\int_0^1 k_r^2\psi(t)\, t^{\frac{1}{2}m} J_m(k_r\sqrt{t})\,dt\Big\}. \tag{11}$$

13. By (10) and (11) we get V_r expressed as the sum of four product terms, of which two are

$$2C_r \frac{\phi(1)\, J_m(k_r)}{k_r^2}\, z^{-\frac{1}{2}m} J_m(k_r\sqrt{z}) \tag{12}$$

and

$$\tfrac{1}{2}C_r z^{-\frac{1}{2}m} J_m(k_r\sqrt{z}) \int_0^1 \psi(t)\, t^{\frac{1}{2}m} J_m(k_r\sqrt{t})\, dt, \tag{13}$$

and the other two are got from these by putting $z = 1$. The series (13), when multiplied by $z^{\frac{1}{2}m}$, becomes the typical term of the Bessel series $\Sigma P_r J_m(k_r\sqrt{t})$ of the function $z^{\frac{1}{2}m}\psi(z)$.

But, as we have already seen (§ 8),

$$\frac{x^m}{m+H} = \Sigma k_r^{-2} C_r J_m(k_r x)\, J_m(k_r).$$

Therefore, putting $x = \sqrt{z}$, we see that (12) is the typical term of a series which converges to

$$4\phi(1)/(m+H),$$

and therefore, combined with the term got from this by putting $z = 1$, gives, after summation, the result zero. Thus, multiplying by $z^{\frac{1}{2}m}$, we have

$$z^{\frac{1}{2}m}\Sigma V_r = \{\Sigma P_r J_m(k_r\sqrt{t}) - P_r J_m(k_r)\}. \tag{14}$$

But, by (10) and (11),

$$P_r J_m(k_r) = \{A_r - \tfrac{1}{2}C_r\,\phi(1)\, J_m(k_r)\}\,\frac{4}{k_r^2} J_m(k_r), \tag{15}$$

and, since $k_r^{-\frac{1}{2}} A_r \to 0$ and $k_r^{-1} C_r$ and $k_r^{\frac{1}{2}} J_m$ are bounded, it follows that the series $\Sigma P_r J_m(k_r)$ converges uniformly. Hence, by (14), since, as we have seen, ΣV_r converges uniformly, the Bessel series $\Sigma P_r J_m(k_r\sqrt{t})$ converges uniformly.

By the Corollary of § 10, it follows that this series is the Bessel series of its sum function, and that this sum function is $z^{\frac{1}{2}m}\psi(z)$, since this latter function is continuous, and $\Sigma P_r J_m(k_r\sqrt{t})$ is its Bessel series.

But if we put $z = 1$, this sum-function is zero; therefore

$$\Sigma P_r J_m(k_r) = 0,$$

and we get, by (14),

$$z^{\frac{1}{2}m}\Sigma V_r = z^{\frac{1}{2}m}\psi(z),$$

or

$$\Sigma V_r = \psi(z),$$

which, as we have already proved that this series converges uniformly, completes the proof of the theorem.

14. From the preceding theorem, together with the Lemma of § 7, we have immediately the following

THEOREM.—*The second integrated series of the Bessel series*

$$\Sigma A_r J_m(k_r z)$$

of $f(z)$ *converges uniformly to* $\int dx \int f(z)\,dx$ *in* $(0 < z < 1)$.

We have also the following special corollary used later on :—

$$\Sigma \int dz \int \sqrt{z}\, A_r J_m(k_r z)\, dz = \int dz \int \sqrt{z}\, f(z)\, dz. \tag{16}$$

15. We now come to the fundamental theorem which enables us to treat Bessel series like Fourier series.

THEOREM.—*The Bessel series*

$$\sum_{r=0}^{\infty} A_r J_m(k_r z), \tag{1}$$

where

$$A_r = \tfrac{1}{2} C_r \int_0^1 f(\sqrt{t})\, J_m(k_r \sqrt{t})\, dt, \tag{4}$$

behaves in respect of convergence, divergence, oscillation, uniform or otherwise, precisely like the Fourier series of $f(z)$, *provided*

(1) *the point or points considered are internal to the completely open interval* $0 < x < 1$;

(2) *the individual terms of the series* (1) *tend to zero, that is*

$$k_r^{-\frac{1}{2}} A_r \to 0 \quad (r \to \infty).$$

To prove this we use the asymptotic expression (viii). We may write this in the form

$$A_r J_m(k_r z) = z^{-\frac{1}{2}} \{u_r + z^{-1} v_r + z^{-2} w_r\} \ldots,$$

where we use the following abbreviations :—

$$\psi_r = -k_r z + \tfrac{1}{4}(2m+1)\pi, \tag{17}$$

$$u_r = \frac{A_r}{\sqrt{k_r}} \sqrt{\frac{2}{\pi}} \cos \psi_r, \tag{18}$$

$$v_r = \frac{A_r}{\sqrt{k_r}} \sqrt{\frac{2}{\pi}}\, \tfrac{1}{2}(m^2 - \tfrac{1}{4})\, k_r^{-1} \sin \psi_r = \tfrac{1}{2}(m^2 - \tfrac{1}{4}) \int u_r dz, \tag{19}$$

$$w_r = -\frac{A_r}{\sqrt{k_r}} \sqrt{\frac{2}{\pi}}\, \frac{1}{2!\ 2^2} (m^2 - \tfrac{1}{4})(m^2 - \tfrac{9}{4})\, k_r^{-2} \left\{ \cos \psi + \frac{\theta(m^2-4)}{6 k_r z} \right\}, \tag{20}$$

where $|\theta| \leqslant 1$.

Thus writing for brevity, symbolically,

$$S_1 = \sum_{r=0}^{\infty} u_r, \qquad S_2 = \sum_{r=0}^{\infty} v_r, \qquad S_3 = \sum_{r=0}^{\infty} w_r,$$

we may write $$\sum_{r=0}^{\infty} A_r J_m(k_r z) = z^{-\frac{1}{2}} \{S_1 + z^{-1} S_2 + z^{-2} S_3\}, \tag{21}$$

this relation being understood not to imply convergence of the series involved, but merely identity, when we replace the infinite series by the corresponding n-th partial summations.

It is, however, at once evident, from the form of w_r, that S_3 is a uniformly convergent series, since $k_r^{-\frac{1}{2}} A_r$ is bounded.*

16. Since, then, we have seen that the second integrated series of the Bessel series $\Sigma A_r J_m(k_r z)$ converges uniformly to the integral of an integral, and S_3 converges uniformly, it follows that, *in the closed interval* $(e \leqslant z \leqslant 1)$, *where* $0 < e$,

$$\Sigma A_r J_m(k_r z) - z^{-\frac{3}{2}} S_3 = z^{-\frac{1}{2}} S_1 + z^{-\frac{3}{2}} S_2,$$

converges uniformly to the integral of an integral, when twice integrated term by term.

17. But, from the form of v_r, S_2, when *once* integrated, converges, like S_3, uniformly, under the same condition as to A_r. Therefore in the closed interval $(e, 1)$, $z^{-\frac{1}{2}} S_1$, integrated *twice*, converges uniformly to the difference between an integral of an integral and the integral of a continuous function, that is to an integral; the same is therefore true of S_1 itself, when twice integrated.

Hence S_2—whose individual term is a multiple of the integral of that of S_1—when *once* integrated, converges uniformly to an integral. The same is therefore true of $z^{-\frac{3}{2}} S_2$ in the closed interval $(e, 1)$.

It follows, since $$z^{-\frac{3}{2}} S_2$$

and $$z^{-\frac{1}{2}} S_1 + z^{-\frac{3}{2}} S_2,$$

when *twice* integrated, each converge to the integral of an integral in $(e, 1)$, that the same is true of $z^{-\frac{1}{2}} S_1$; and therefore also S_1 *when integrated twice converges uniformly in* $(e, 1)$ *to the integral of an integral.*

We shall see that we can prove more than this, and that S_1 when once integrated converges uniformly to an integral.

* It suffices here that $A_r k_r^{-s}$ is bounded, with $s < \frac{3}{2}$.

18. At this point, for reasons which will appear subsequently, we shall make the assumption that m is rational with μ for denominator. It must not, however, be supposed that there is any such restriction involved in the theorem itself.

By (3) we may then write

$$k_r = \frac{\pi}{4\mu}(R+\beta_r), \tag{22}$$

where R is an integer which increases by 4μ each time r increases by 1, and

$$\beta_r = \beta R^{-1} + O(R^{-2}). \tag{23}$$

Writing ϵ_1 and ϵ_2 for the sine and cosine of $\frac{1}{4}(2m+1)\pi$, we then have, by (17), putting

$$y = \pi z/4\mu, \tag{24}$$

$$\begin{aligned}\cos\psi_r &= \epsilon_1 \cos k_r z + \epsilon_2 \sin k_r z \\ &= (\epsilon_1 \cos Ry + \epsilon_2 \sin Ry)\cos\beta_r y + (\epsilon_2 \cos Ry - \epsilon_1 \sin Ry)\sin\beta_r y.\end{aligned}$$

Thus, writing

$$\tau_r = \sqrt{\frac{2}{\pi k_r}} A_r(\epsilon_1 \cos Ry + \epsilon_2 \sin Ry), \tag{25}$$

$$\tau_{1,r} = \sqrt{\frac{2}{\pi k_r}} A_r(\epsilon_2 \cos Ry - \epsilon_1 \sin Ry), \tag{26}$$

$$T = \sum_{r=0}^{\infty} \tau_r, \tag{27}$$

$$T_1 = \sum_{r=1}^{\infty} R^{-1}\tau_{1,r} = -\Sigma\int \tau_r\, dy, \tag{28}$$

we get

$$\begin{aligned}S_1 = \sum_{r=0}^{\infty} u_r &= \sum_{r=0}^{\infty} \sqrt{\frac{2}{\pi k_r}} A_r \cos\psi_r = \sum_{r=0}^{\infty} (\tau_r \cos\beta_r y + \tau_{1,r}\sin\beta_r y) \\ &= T + \beta y T_1 + T_2,\end{aligned} \tag{29}$$

where T_2 is a uniformly convergent series whose sum, as we shall next show, is an integral.

19. Indeed, since τ_r and $\tau_{1,r}$ are bounded, and

$$\frac{d}{dy}\tau_r = R\tau_{1,r},$$

$$\frac{d}{dy}\tau_{1,r} = -R\tau_r,$$

we have $$\frac{d}{dy}(u_r-\tau_r-\beta_r\tau_{1,r}\,y/R+\tfrac{1}{2}\beta^2\tau_r y^2/R^2)$$

$$= -\frac{d}{dy}\{\tau_r(1-\tfrac{1}{2}\beta^2y^2/R^2-\cos\beta_r y)+\tau_{1,r}(\beta_r y/R-\sin\beta_r y)\}$$

$$= O(R^{-2}).$$

Thus $$\Sigma\frac{d}{dy}(u_r-\tau_r-\beta_r\tau_{1,r}\,y/R+\tfrac{1}{2}\beta^2\tau_r y^2/R^2),$$

is a uniformly convergent series. Integrating this term-by-term, we see that

$$S-T-\beta_r yT_1+\tfrac{1}{2}\beta^2y^2\Sigma\tau_r/R^2 = T_2+\tfrac{1}{2}\beta^2y^2\Sigma\tau_r/R^2$$

converges uniformly to an integral.

Since $\Sigma\tau_r/R^2$ is the first integrated series of T_1, we shall see by the next article that this proves the statement made at the end of the last article.

20. Now by (28) and (26) we see that T_1 is the Fourier series of a function of y whose square is absolutely integrable, since the sum of the squares of its coefficients is $\frac{2}{\pi}\sum_{r=0}^{\infty}(k_r^{-\frac{1}{2}}A_rR^{-1})^2$, which is convergent since

$$k_r^{-\frac{1}{2}}A_r\to 0.*$$

Therefore the first integrated series of T_1 converges uniformly, in the present circumstances, to an integral.

21. Integrating twice term-by-term the series

$$S_1-\beta yT_1-T_2,$$

we get therefore, by §§ 17, 20, 18 and 19, a uniformly convergent series whose sum is an integral. That is, since, by (29), each term of this series is equal to the corresponding term of T, *the second integrated series of T converges uniformly to an integral.*

But T is, by (27) and (25), a trigonometrical series whose coefficients tend to zero, since $k_r^{-\frac{1}{2}}A_r\to 0$ $(r\to\infty)$. Thus T is an R. F. series in the interval $(\epsilon\leqslant z\leqslant 1)$, that is in $(\pi\epsilon/4\mu\leqslant y\leqslant\pi/4\mu)$. Since ϵ is as small as we please, this proves that *T is an R. F. series in the interval $(0\leqslant y\leqslant\pi/4\mu)$, m being rational with μ for denominator.*

---

* This would still be the case of course if $A_rk_r^{-s}$ were bounded, where $s<\frac{3}{2}$.

Hence in this interval T_1, the first integrated series of T_1, converges to an integral, the convergence being uniform in every closed interval inside the interval of restriction.

Thus we have proved that

$$S_1 = T+\beta T_1 y+T_2, \tag{29}$$

where T is an R. F. series in $(0 < y \leqslant \pi/4m)$, and T_1 is the integrated series of T, and is accordingly a Fourier series, which in $(0 < y \leqslant \pi/4m)$ converges uniformly to an integral, while T_2 is a uniformly convergent series, whose sum is an integral.

Also we had

$$\sum_{r=0}^{\infty} A_r J_m(k_r z) = z^{-\frac{1}{2}}(S_1+z^{-1}S_2+z^{-2}S_3), \tag{21}$$

where a constant multiple of S_2 is the integrated series of S_1, and therefore S_2 converges uniformly to an integral in every closed interval in $(0 < y \leqslant \pi/4m)$ or $(0 < z \leqslant 1)$, and S_3 is a uniformly convergent series whose sum, as we shall now prove, is an integral.

22. In fact, by (19) and (20),

$$w_r = (m^2-\tfrac{9}{4})\frac{1}{2!}\int v_r dz+A_r Q_m(k_r z), \tag{22}$$

where, by the formula (viii),

$$Q_m(x) = \sqrt{\left(\frac{\pi x}{2}\right)} J_m(x)-\cos\psi-\frac{m^2-\frac{1}{4}}{2x}\sin\psi+\frac{(m^2-\frac{1}{4})(m^2-\frac{9}{4})}{2!\,(2x)^2}\cos\psi. \tag{23}$$

It is therefore only necessary to shew that $\sum_{r=0}^{\infty} A_r k_r^{-\frac{1}{2}} Q_m(k_r z)$ converges uniformly to an integral, since the first term on the right in (22), when summed, converges uniformly to a numerical multiple of the integral of the sum-function of S_2, *i.e.* to the integral of an integral.

Now, differentiating (23), we see that $\Sigma A_r k_r^{-\frac{1}{2}} \frac{d}{dx} Q_m(k_r z)$ is a uniformly convergent series*; therefore, integrating term-by-term, $\Sigma A_r k_r^{-\frac{1}{2}} Q_m(k_r z)$ converges uniformly to an integral.

Hence, as stated, the sum of the series S_3 is an integral.

* It is only necessary that $A_r k_r^{-s}$ should be bounded, with $s < \frac{5}{2}$.

23. Thus, finally,

$$\Sigma A_r J_m(k_r z) = z^{-\frac{1}{2}}(S_1 + z^{-1} S_2 + z^{-2} S_3), \tag{21}$$

where S_2 and S_3 are uniformly convergent series, whose sums are integrals, and

$$S_1 = T + \beta T_1 y + T_2, \tag{29}$$

where T_1 and T_2 are uniformly convergent series whose sums are integrals, while T is an R. F. series in $(0 < y \leqslant \pi/4\mu)$, where $y = z/4\mu$.

It follows that $\Sigma A_r J_m(k_r z)$ converges, or diverges, according as this R. F. series does so. Moreover the relations (21) and (29) are identities when we replace each of the infinite series involved by the corresponding n-th partial summations, so that, even when the R. F. series and the Bessel series do not converge, their oscillation properties are identical, and in particular uniform convergence or oscillation, or bounded convergence or oscillation, of the one series entails the same property in the other. Also term-by-term integration properties of the one series may be immediately inferred for one of the series if they are known to hold for the other.

24. Thus we may write

$$\Sigma A_r J_m(k_r z) = z^{-\frac{1}{2}} \{T + \phi(z)\}, \tag{30}$$

where $\phi(z)$ is an integral.

Integrating twice after multiplying by $\sqrt{z}$, and denoting the function associated with T by $g(y)$,

$$\Sigma \int dz \int \sqrt{z}\, f(z)\, dz = \int dz \int g\left(\frac{\pi z}{4\mu}\right) dz + \int dz \int \phi(z)\, dz,$$

whence, differentiating twice at any internal point of the interval $(e, 1)$,

$$g\left(\frac{\pi z}{4\mu}\right) = \sqrt{z}\, f(z) - \phi(z),$$

or

$$g(y) = \sqrt{\left(\frac{4\mu y}{\pi}\right)} f\left(\frac{4\mu y}{\pi}\right) - \phi\left(\frac{4\mu y}{\pi}\right).$$

But the Fourier series of $\sqrt{z}\, f(z)$ converges or diverges at all points of $(e, 1)$ according as that of $f(z)$ does so. Thus our R. F. series converges or diverges according as the Fourier series of $f(z)$ does so.

Thus, finally, the Bessel series $\Sigma A_r J_m(k_r z)$, whose associated function is $f(z)$, converges, diverges, or oscillates, precisely like the Fourier series of $f(z)$.

This proves the theorem enunciated in § 15, when m is rational.

25.* The above was proved on the supposition that m was a rational number. This is not necessary, but was convenient, since it enabled us to use the existing mathematical machinery. We shall now shew that the result is the same, and the general trend of the reasoning also, when m is irrational.

In this case, writing $k = \frac{1}{4}(2m+1+4s)$,

$$\pi z = \pi - y,$$

$$\beta_r = \frac{\pi a(r)}{r} = \frac{\beta}{r} + O(r^{-2}), \tag{23a}$$

we have $$\psi_r = (k+r)\,y - (r+s)\,\pi - \beta_r z. \tag{17a}$$

Therefore, putting $$\tau_r = (-)^{r+s} A_r \sqrt{\left(\frac{2}{\pi k_r}\right)} \cos(k+r)y, \tag{25a}$$

$$\tau_{1,r} = (-)^{r+s} A_r \sqrt{\left(\frac{2}{\pi k_r}\right)} \sin(k+r)y, \tag{26a}$$

and using T, T_1, and S_1 in the same sense as before, we have, as in § 18,

$$S_1 = \sum_0^\infty u_r = \sum_0^\infty \sqrt{\left(\frac{2}{\pi k_r}\right)} A_r \cos\psi_r = T + \beta T_1 z + T_2. \tag{29}$$

The argument from this point is unaltered in character till we arrive at § 21 : for, though T_1 is no longer a Fourier series, it is the sum of two Fourier series, multiplied respectively by $\cos ky$ and $\sin ky$. Thus it is still true that the first integrated series of T_1 converges uniformly to an integral.

26. We thus see, as in § 21, that the second integrated series of T converges uniformly to the integral of an integral. But T is not any longer an R. F. series.

We proceed therefore to make a short digression on a non-harmonic trigonometrical series

$$\sum_{r=0}^\infty \{a_r \cos(k+r)\,x + b_r \sin(k+r)\,x\},$$

whose individual term tends to zero, while its second integrated series converges uniformly to the integral of an integral.

---

* Modified January 29th, 1918.

27. Let the series

$$-\sum_{r=0}^{\infty}(r+k)^{-2}\{a_r\cos(r+k)x+b_r\sin(r+k)x\} \quad (0<k<1),$$

where

$$\left.\begin{aligned} a_n &\to 0,\\ b_n &\to 0,\end{aligned}\right\} \quad (n\to\infty),$$

converge uniformly, and in the interval $(\alpha \leqslant x \leqslant \beta)$ let the sum-function $G(x)$ be a repeated integral

$$\int dx\int f(x)\,dx.$$

Then

$$\tfrac{1}{2}\{G(x+t)+G(x-t)\} = -\sum_{r=0}^{\infty}\frac{\cos(r+k)t}{(r+k)^2}\{a_r\cos(r+k)x+b_r\sin(r+k)x\}$$

$$= -\sum_{r=0}^{\infty}(r+k)^{-2}A_r\cos(r+k)t, \qquad (31)$$

and, the convergence being uniform,

$$\int_0^{\infty}\tfrac{1}{2}\{G(x+t)+G(x-t)\}\frac{1-\cos t}{t^2}\,dt$$

$$= -\sum_{r=0}^{\infty}(r+k)^{-2}A_r\int_0^{\infty}\frac{1-\cos t}{t^2}\cos(k+r)t\,dt. \qquad (32)$$

Now

$$2(1-\cos t)\cos(k+r)t = 2\cos(k+r)t-\cos(k+r+1)t-\cos(k+r-1)t.$$

Therefore, since $$\int_0^{\infty}\frac{1-\cos qt}{t^2}dt = \frac{\pi}{2}|q|,$$

the integral under the sign of summation in (32) is zero if $r \geqslant 1$, and when $r=0$ has the value

$$\tfrac{1}{2}k^{-2}A_0\frac{\pi}{2}(2-2k) = \frac{\pi}{2}k^{-2}A_0(1-k).$$

Thus $$\frac{2}{\pi}\int_0^{\infty}\tfrac{1}{2}\{G(x+t)+G(x-t)\}\frac{1-\cos t}{t^2}\,dt = -k^{-2}A_0(1-k). \qquad (33)$$

Again, similarly,

$$\int_0^{\infty}\tfrac{1}{2}\{G(x+t)+G(x-t)\}\cos t\,\frac{1-\cos t}{t^2}\,dt$$

$$= -\sum_{r=0}^{\infty}\tfrac{1}{2}(r+k)^{-2}A_r\int_0^{\infty}2\cos t\cos(k+r)t\,\frac{1-\cos t}{t^2}\,dt. \qquad (34)$$

Now

$$2\cos(k+r)\,t\cos t-2\cos(k+r)\,t\cos^2 t$$

$$=\cos(k+r+1)\,t+\cos(k+r-1)\,t-\cos(k+r)\,t-\cos(k+r)\,t\cos 2t$$

$$=\cos(k+r+1)\,t+\cos(k+r-1)\,t-\cos(k+r)\,t-\tfrac{1}{2}\cos(k+r+2)\,t$$

$$-\tfrac{1}{2}\cos(k+r-2)\,t\,;$$

whence, as before, all the integrals on the right of (34) vanish except for $r=0$, $r=1$, and we get

$$\frac{2}{\pi}\int_0^\infty \{G(x+t)+G(x-t)\}\cos t\,\frac{1-\cos t}{t^2}\,dt=-k^{-2}A_0k-(k+1)^{-2}A_1(1-k). \tag{35}$$

For all values of $n>1$, we have then, in like manner,

$$\int_0^\infty \tfrac{1}{2}\{G(x+t)+G(x-t)\}\cos nt\,\frac{1-\cos t}{t^2}\,dt$$

$$=-\sum_{r=0}^{\infty}(r+k)^{-2}A_r\int_0^\infty \cos(r+k)\,t\cos nt\,\frac{1-\cos t}{t^2}\,dt. \tag{36}$$

Now since

$$2\cos(r+k)\,t\cos nt\,(1-\cos t)$$

$$=\cos(r+k+n)\,t+\cos(r+k-n)\,t-\cos(r+k)\,t\cos(n+1)\,t$$

$$-\cos(r+k)t\cos(n-1)\,t$$

$$=\cos(r+k+n)\,t+\cos(r+k-n)\,t-\tfrac{1}{2}\cos(r+k+n+1)\,t$$

$$-\tfrac{1}{2}\cos(r+k+n-1)\,t-\tfrac{1}{2}\cos(r+k-n+1)\,t-\tfrac{1}{2}\cos(r+k-n-1)\,t,$$

the integrals on the right of (36) all vanish except for $r=n-1$, when the value is $\frac{1}{4}\pi k$, and for $r=n$, when it is $\frac{1}{4}\pi(1-k)$.

Thus, for $n>1$,

$$\frac{2}{\pi}\int_0^\infty \{G(x+t)+G(x-t)\}\cos nt\,\frac{1-\cos t}{t^2}\,dt$$

$$=-(k+n-1)^{-2}A_{n-1}k-(k+n)^{-2}A_n(1-k),$$

or, differentiating twice,

$$\frac{2}{\pi}\frac{d^2}{dx^2}\int_0^\infty \{G(x+t)+G(x-t)\}\cos nt\,\frac{1-\cos t}{t^2}\,dt=kA_{n-1}+(1-k)A_n. \tag{37}$$

Summing, and taking account of (35) and (33), we get

$$\frac{2}{\pi}\frac{d^2}{dx^2}\int_0^\infty \{G(x+t)+G(x-t)\}\,\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\sin(n+\tfrac{1}{2})\,t\,\frac{1-\cos t}{t^2}\,dt$$

$$= \sum_{r=0}^{n} A_r + kA_n. \quad (38)$$

28.* Writing

$$\operatorname{cosec}\tfrac{1}{2}\sin(n+\tfrac{1}{2})t = \sin nt\cot\tfrac{1}{2}t - \cos nt,$$

we have the left-hand side of (38) expressed as the sum of two terms, the second of which, being by (37) equal to $kA_{n-1}+(1-k)A_n$, tends to zero as $n\to\infty$. Thus we are left with the first term, which we write $4I_n/\pi$, where

$$I_n = \frac{d^2}{dx^2}\int_0^\infty \tfrac{1}{2}\{G(x+t)+G(x-t)\}\sin nt\cot\tfrac{1}{2}t\,\frac{1-\cos t}{t^2}\,dt$$

$$= \frac{d^2}{dx^2}\int_0^\infty \tfrac{1}{2}\{G(x+t)+G(x-t)\}\,t^{-2}\sin t\sin nt\,dt.$$

We now divide the interval $(0,\infty)$ up into $(0,2\pi)$, $(2\pi,4\pi)$, ..., and sum the corresponding integrals, using the formula

$$\sin 2k\pi\,.\,\tfrac{1}{2}\{G[x+(2p\pi+t)]+G[x-(2p\pi+t)]\}$$

$$= -\sum_{r=1}^{\infty}(r+k)^{-2}A_r\cos\{(r+k)(2p\pi+t)\}\sin 2k\pi$$

$$= \cos 2kp\pi\sin 2k\pi\,.\,\tfrac{1}{2}\{G(x+t)+G(x-t)\}$$

$$-\sin 2kp\pi\sin 2k\pi\,.\,\tfrac{1}{2}\{G(x+t)-G(x-t)\}$$

$$= \sin 2k(p+1)\pi\,.\,\tfrac{1}{2}\{G(x+t)+G(x-t)\}$$

$$-\sin 2kp\pi\,.\,\tfrac{1}{2}\{G[x+(2\pi-t)]+G[x-(2\pi-t)]\}.$$

Hence

$$I_n = \frac{d^2}{dx^2}\Big[\int_0^{2\pi}\tfrac{1}{2}\{G(x+t)+G(x-t)\}\,\chi_1(t)\sin t\sin nt\,dt$$

$$-\int_0^{2\pi}\tfrac{1}{2}\{G[x+(2\pi-t)]+G[x-(2\pi-t)]\}\,\chi_2(2\pi-t)\sin t\sin nt\,dt\Big]$$

$$= \frac{d^2}{dx^2}\int_0^{2\pi}\tfrac{1}{2}\{G(x+t)+G(x-t)\}\{\chi_1(t)-\chi_2(t)\}\sin t\sin nt\,dt,$$

---

* Modified January 29th, 1918.

where $$\chi_1(t) = t^{-2} + \operatorname{cosec} 2k\pi \sum_{p=1}^{\infty} (2p\pi + t)^{-2} \sin 2k(p+1)\pi,$$

$$\chi_2(2\pi - t) = \operatorname{cosec} 2k\pi \sum_{p=1}^{\infty} (2p\pi + t)^{-2} \sin 2kp\pi.$$

Thus $$\chi_1(2\pi) = \chi_2(2\pi).$$

Hence if we define a function $\lambda(t)$ as equal to $\chi_1(t) - \chi_2(t)$ from e to 2π, we shall have $\lambda(2\pi) = 0$, and $\lambda(t)$ is the integral of an integral in this interval. If therefore we assign suitable values to $\lambda(t)$ in $(0, e)$, this function will be the integral of an integral in $(0, 2\pi)$, and have zero for its values at 0 and 2π. Then $\lambda(t) \sin t$ will be the integral of an integral in $(0, 2\pi)$, with a zero value and a zero differential coefficient at each of the end-points 0 and 2π.

By a theorem which I have recently proved,* we have therefore uniformly,

$$\frac{d^2}{dx^2} \int_0^{2\pi} \tfrac{1}{2} \{G(x+t) + G(x-t)\} \lambda(t) \sin t \sin nt\, dt \to 0 \quad (n \to \infty).$$

Hence we may replace I_n by

$$\frac{d^2}{dx^2} \int_0^{2\pi} \tfrac{1}{2} \{G(x+t) + G(x-t)\} \{\chi_1(t) - \chi_2(t) - \lambda(t)\} \sin t \sin nt\, dt$$

$$= \frac{d^2}{dx^2} \int_0^{e} \tfrac{1}{2} \{G(x+t) + G(x-t)\} \{\chi_1(t) - \chi_2(t) - \lambda(t)\} \sin t \sin nt\, dt$$

$$= \int_0^{e} \tfrac{1}{2} \{f(x+t) + f(x-t)\} \{\chi_1(t) - \chi_2(t) - \lambda(t)\} \sin t \sin nt\, dt,$$

since $G(t)$ is the integral of an integral in $(x-e, x+e)$ and $f(t) = d^2G/dt^2$.

Now $$\chi_1(t) - \chi_2(t) - \lambda(t) - t^{-2}$$

is a bounded function of t in $(0, e)$; therefore

$$\tfrac{1}{2} \{f(x+t) + f(x-t)\} \{\chi_1(t) - \chi_2(t) - \lambda(t) - t^{-2}\} \sin t$$

is a summable function in $(0, e)$. Hence when we multiply by $\sin nt$ and integrate from 0 to e, the integral tends uniformly to zero by the theorem of Riemann-Lebesgue. We may therefore replace I_n by

$$\int_0^{e} \tfrac{1}{2} \{f(x+t) + f(x-t)\} t^{-2} \sin t \sin nt\, dt.$$

---

* "On Non-Harmonic Trigonometrical Series," 1917, *Roy. Soc. Proc.* (A), Vol. 95, pp. 237-249.

29. Hence we have

$$\frac{1}{\pi}\int_0^e \{f(x+t)+f(x-t)\}\sin nt\cot\tfrac{1}{2}t\,\frac{1-\cos t}{t^2}\,dt+\eta_n=\sum_{r=0}^{n}A_r,$$

where $\eta_n\to 0$ uniformly.

Let us now write $\dfrac{1-\cos t}{t^2}=\tfrac{1}{2}+t^2\dfrac{1-\frac{1}{2}t^2-\cos t}{t^4}$.

Then since $$t^2\cot\tfrac{1}{2}t\,\frac{1-\frac{1}{2}t^2-\cos t}{t^4}$$

is a bounded function in $(0, e)$, the part of our integral involving this factor in the integrand tends to zero by the theorem of Riemann-Lebesgue.

Thus, finally,

$$\frac{1}{2\pi}\int_0^e \{f(x+t)+f(x-t)\}\sin nt\cot\tfrac{1}{2}t\,dt+\eta_n = \sum_{r=0}^{n}\{a_r\cos(r+k)\,x+b_r\sin(r+k)\,x\}, \quad (39)$$

where, as $n\to\infty$, $\eta_n\to 0$ uniformly.

30. The expression here found for the partial summation of our non-harmonic trigonometrical series is, barring an additive term which tends to zero as $n\to\infty$, the same as the partial summation of the Fourier series of $f(x)$. Indeed in the paper quoted this form of the partial summation is easily obtained [see equations (6) and (9) of that paper].

We may accordingly write in the more familiar expression for the partial summation of the Fourier series, and get

$$\frac{1}{2\pi}\int_0^e \{f(x+t)+f(x-t)\}\sin(n+\tfrac{1}{2})\,t\,\operatorname{cosec}\tfrac{1}{2}t\,dt+\eta_n = \sum_{r=0}^{n}\{a_r\cos(r+k)\,x+b_r\cos(r+k)\,x\}, \quad (40)$$

where $\eta_n\to 0$ uniformly.

Thus we see that, at a point x where $\alpha<x<\beta$, the upper and lower functions of the non-harmonic trigonometrical series

$$\sum_{r=0}^{\infty}\{a_r\cos(r+k)\,x+b_r\sin(r+k)\,x\},$$

where (i) $a_r\to 0, \quad b_r\to 0,$

and (ii) *the sum of the second integrated series is in* (α, β) *the integral of*

the integral of a summable function $f(x)$, are the same as those of the Fourier series of $f(x)$, and the difference of corresponding partial summations tends uniformly to zero.

31. Using this result in place of the corresponding theorem in the theory of R. F. series, the theorem of § 15 follows as before, m being now not necessarily rational.

Thus the fundamental theorem of § 15 has been established in all its generality.

More accurately what we have proved may be stated in the following form :—

Theorem.—*The upper and lower functions of the Bessel series of $f(x)$, at a point x internal to the interval $0 < x < 1$, are equal to those of the Fourier series of $f(x)$, and the corresponding partial summations differ only by a quantity which tends uniformly to zero, provided the individual terms of the Bessel series tend to zero.*

32. We have immediately the following corollaries :—

Cor. 1.—*The first integrated series*

$$\sum_{r=0}^{\infty} \int A_r J_r(k_r z)\, dz$$

of a Bessel series whose individual terms tend to zero, that is

$$k^{-\frac{1}{2}} A_r \to 0 \quad (r \to \infty),$$

converges uniformly to an integral at every point and in every interval in $(0 < z < 1)$.

For, in equation (30), we have seen that the integrated series of T converges uniformly to an integral in any closed interval inside $(0<z<1)$. Hence, multiplying by $z^{-\frac{1}{2}}$ and integrating (30) term-by-term, which is allowable by reason of the uniform convergence, the required result appears immediately.

33. The following extension of the main theorem is important :—

Cor. 2.—*The theorem is still true for a series of Bessel functions of the form* (1), *whose individual terms tend to zero, that is for any series*

$$\Sigma A_r J_m(k_r z), \tag{1}$$

where

$$k_r^{-\frac{1}{2}} A_r \to 0 \quad (r \to \infty),$$

provided in some interval containing the point, or points, under consideration the sum of the second integrated series is the integral of an integral.

In fact the condition (4), which makes our series in § 15 a Bessel series, nowhere enters into the proof itself of the main theorem, except tacitly, in quoting the theorem that the second integrated series converges to a repeated integral.

Hence, as in Cor. 1, *the first integrated series of such a series as we here contemplate converges to an integral.*

The function associated with the series of Bessel functions will then be the differential coefficient of the sum-function of the integrated series, in the interval under consideration in which this latter function is an integral.

It is evidently sufficient for the fulfilment of the conditions of the above theorem, if

$$k_r^{-\frac{1}{2}} A_r \to 0 \quad (r \to \infty),$$

and the integrated series converges *uniformly* to an integral.

34. It is unnecessary to enumerate all the results for Bessel series and series of Bessel functions which can now be transferred wholesale from the theory of Fourier series to that of these series. These theorems include, beside sufficient conditions for convergence, uniform convergence or oscillation, bounded convergence or oscillation, and so forth, as well as for Cesàro convergence, all theorems involving term-by-term integration when multiplied by suitable functions, provided we remain in the interior of the completely open interval (0, 1).

As an example we may quote one of the simplest of those theorems which are an immediate consequence of the main theorem :—

The Bessel series of a function which, in an interval (α, β), *is a continuous function of bounded variation converges uniformly to that function in* (α, β); *in particular this is the case in any interval* (α, β) *in which the function is an integral.*

Hence we have the following important result :—

THEOREM.—*If* $f(x)$ *is, in a neighbourhood surrounding the point* x, *the integral of a function whose Fourier series converges at that point, e.g. if it is the integral of a continuous function of bounded variation in such a neighbourhood, then the derived series of the Bessel series of* $f(x)$ *will converge at the point in question to the value* $f'(x)$.

For let the series be $\Sigma A_r J_m(k_r x)$,

where $$A_r = C_r \int_0^1 t f(t) J_m(k_r t)\, dt,$$

$f(t)$ being an integral in (α, β).

Then, by the formula (vi), if $m \geqslant 1$,

$$k_r \int_0^x t^m J_{m-1}(k_r t)\, dt = x^m J_m(k_r x).$$

Thus the series $\Sigma k_r A_r J_{m-1}(k_r x)$

is a series of Bessel functions, which, when multiplied by x^m and integrated term-by-term, converges uniformly to an integral, namely to $x^m f(x)$.

Therefore the integrated series of this series converges uniformly to $f(x)$.

If therefore the condition $\sqrt{k_r} A_r \to 0 \quad (r \to \infty)$

be satisfied, we can apply our theorem, and the theorem as enunciated at once follows. It will be seen by the following theorem that this condition is satisfied, which proves the required result.

35. Now, if $f(x)$ is an integral, $f'(x)$ is summable, and therefore in any interval in which $f(x)$ is an integral, and which does not contain the origin, $\sqrt{x} f'(x) - m f(x)/\sqrt{x}$ is summable, and $x^{m+1} f(x) \to 0$ as $x \to 0$. Thus we see that the result required to complete the preceding proof is a special case of the following theorem.

Theorem. — *If*

(i) $x^{m+\frac{1}{2}}\, d/dx \{x^{-m} f(x)\}$ *is summable in* $(0, 1)$,

and (ii) $x^{m+1} f(x) \to 0 \quad (x \to 0)$,

then $$\int_0^1 J_m(k_r x)\, x f(x)\, dx = O(k_r^{-\frac{3}{2}}).$$

Let us write $g(x)$ for the summable function involved in the hypothesis (i).

Then, by the formula (vi),

$$k_r \int_\epsilon^1 J_m(k_r x)\, x f(x)\, dx$$

$$= \int_\epsilon^1 x^{-m} f(x)\, d \int_0^x k_r x^{m+1} J_m(k_r x)\, dx$$

$$= f(1)\, J_{m+1}(k_r) - f(\epsilon)\, J_{m+1}(k_r \epsilon) + \int_\epsilon^1 x^{m+1} J_{m+1}(k_r x)\, \frac{d}{dx} \{x^{-m} f(x)\}\, dx$$

$$= f(1)\, \frac{m+H}{k_r}\, J_m(k_r) - f(\epsilon)\, J_{m+1}(k_r \epsilon) + \int_\epsilon^1 g(x) x^{\frac{1}{2}} J_{m+1}(k_r x)\, dx.$$

Now $x^{\frac{1}{2}} J_{m+1}(k_r x)$ is bounded in (0, 1) and $g(x)$ is summable. Therefore, if we let $\epsilon \to 0$, the integral on the right will exist up to and including the limit. The same is true of the integral on the left. The term $f(\epsilon)\, J_{m+1}(k_r \epsilon)$ will therefore also have a unique and finite limit, and, by the hypothesis (ii), this limit will be zero.

Thus we get

$$k_r \int_0^1 J_m(k_r x)\, x f(x)\, dx = f(1)\, \frac{m+H}{k^{\frac{3}{2}}} \{\sqrt{k_r}\, J_m(k_r)\} + \int_0^1 g(x)\, x^{\frac{1}{2}} J_{m+1}(k_r x)\, dx.$$

But
$$J_{m+1}(k_r x) = \frac{m}{k_r x} J_m(k_r x) - J'_m(k_r x).$$

Therefore

$$\sqrt{(\tfrac{1}{2}\pi k_r x)}\, J_{m+1}(k_r x)$$

$$= \sqrt{(\tfrac{1}{2}\pi k_r x)} \{(m-\tfrac{1}{2})\, J_m(k_r x)/k_r x - J'_m(k_r x) - J_m(k_r x)/2k_r x\}$$

$$= -\sin \{\tfrac{1}{4}(2m+1)\, \pi - k_r x\} + O(k_r^{-1}),$$

by formulæ (viii) and (ix). Therefore

$$\int_\epsilon^1 g(x)\, x^{\frac{1}{2}} J_{m+1}(k_r x)\, dx = -k_r^{-\frac{1}{2}} \int_\epsilon^1 g(x) \sin \{\tfrac{1}{4}(2m+1)\, \pi - k_r x\}\, dx + O(k_r^{-\frac{3}{2}}),$$

where the integral on the right vanishes, when $r \to \infty$, by the theorem of Riemann-Lebesgue. Therefore

$$\int_0^1 g(x)\, x^{\frac{1}{2}} J_{m+1}(k_r x)\, dx = k^{-\frac{1}{2}} (\eta_\epsilon + \eta_r),$$

where η_ϵ is as small as we please, provided ϵ is as small as is convenient, and η_r is as small as we please, if r is sufficiently large.

We thus get

$$\int_0^1 J_m(k_r x)\, x f(x)\, dx = O(k^{-\frac{1}{2}}) + k^{-\frac{3}{2}}(\eta_\epsilon + \eta_r)$$
$$= o(k^{-\frac{3}{2}}),$$

which proves our theorem.

§§ 36–38.

Appendix on the Convergence of the Series $\Sigma A_r J_m(k_r x)$ *at the Point* $x = 1$.

36. We state first three auxiliary theorems:—

THEOREM 1.—*The series*

$$x^{\frac{1}{2}} \Sigma C_r J_m(k_r x)\, J_m(k_r)$$

oscillates boundedly for all values of x *in the interval* $(0, 1-\epsilon)$.

[For $\eta < x < 1-\epsilon$, this theorem is obvious, using the asymptotic expression (viii) for $J_m(k_r x)$.]

THEOREM 2.—*The series*

$$\Sigma k_r^{-1} C_r J_{m+1}(k_r x)\, J_m(k_r)$$

converges boundedly for all values of x *in* $(1-\epsilon, 1)$, *converges uniformly to zero in the interval* $(\eta, 1-\epsilon)$, *and converges to unity at the point* $x = 1$.

THEOREM 3.—*The series*

$$\frac{1}{H-m} \Sigma C_r \frac{J_m(k_r x)\, J_{m+1}(k_r)}{k_r} = \Sigma C_r \frac{J_m(k_r x)\, J_m(k_r)}{k_r^2}$$

converges uniformly for all values of x *in* $(0, 1)$ *to the value* $x^m/(H-m)$.

Theorem 3 has been already proved in § 8.

To prove the other two theorems we have only to slightly modify the expression to be considered, and the procedure to be followed.

Thus to prove Theorem 1, we have plainly to introduce t^2 into the numerator of the integrand of our complex integral. The sum of n terms of our series in Theorem 1 is accordingly the value of

$$\frac{1}{2\pi i} \int \frac{t J_m(xt)}{t J'_m(t) + H J_m(t)}\, dt = \frac{1}{2\pi i} \int Q(t)\, dt,$$

say, taken round a contour which encloses the roots $\pm k_r$, ($r = 1$ to $r = n$) of the equation

$$\frac{tJ'_m(t)}{J_m(t)} + H = 0.$$

As we only require to prove this sum a bounded function of n and x, in the interval $(0 \leqslant x \leqslant 1-\epsilon)$, we shall omit numerical factors whenever convenient. Take as the contour a rectangle, containing the origin, of which the corners* are the four points $\pm\sigma \pm i\tau$. Then since $Q(t) = Q(-t)$, and $Q(u+iv)$ and $Q(u-iv)$ agree in their real parts, we see that

$$\Sigma J_m(k_r x)\, J_m(k_r) = \mathfrak{R}\left[\frac{1}{2\pi}\int_0^\tau Q(\sigma+it)\,dt - \frac{1}{2\pi i}\int_0^\sigma Q(s+i\tau)\,ds\right].$$

We choose $$\sigma = \kappa + n\pi - \tfrac{1}{2}\pi,$$

n being so large that $n^{-1}\alpha(n)$, in the expression $\kappa + n\pi + n^{-1}\alpha(n)$ for k_n/π, is numerically less than $\frac{1}{2}\pi$.

We then get, putting $\beta = \sigma + it$,

$$(2\pi\beta)^{\frac{1}{2}}\left\{J'_m(\beta) + \frac{H}{\beta}J_m(\beta)\right\} > \tfrac{1}{2}e^t. \tag{1}$$

If, on the other hand, we take $\beta = s + i\tau$, where τ is fixed and sufficiently great, and s varies from 0 to σ, we have

$$(2\pi\beta)^{\frac{1}{2}}\left\{J'_m(\beta) + \frac{H}{\beta}J_m(\beta)\right\} > \tfrac{1}{4}e^\tau. \tag{2}$$

But $$|\beta^{\frac{1}{2}}x^{\frac{1}{2}}J_m(\beta x)| < Ne^{t\alpha},$$

where $\beta = s+it$, N is independent of β and x, and $0 \leqslant x \leqslant 1$. Hence

$$|Q(\sigma+it)| < 2Ne^{tx}/e^t,$$

and therefore

$$\left|x^{\frac{1}{2}}\int_0^\tau Q(\sigma+it)\right| < 2N\int_0^\tau e^{-t(1-x)}\,dt < 2N\,\frac{1-e^{-\tau(1-x)}}{1-x},$$

* We here follow Hobson, *loc. cit.* But we require only a special case of the general result proved by him.

and is therefore bounded when we make τ sufficiently large. Similarly

$$| Q(s+i\tau) | < 4Ne^{\tau x}/e^{\tau} < 4Ne^{-\tau(1-x)},$$

and

$$x^{\frac{1}{2}} \left| \int_0^{\sigma} Q(s+i\tau)\, ds \right| < 4N\sigma e^{-\tau(1-x)}.$$

Now the roots k_r are all on the real axis. Hence there is nothing to prevent our always choosing our contour so that $\sigma e^{-\tau(1-x)}$ is as small as we please, without altering the roots enclosed.

It is plain then that Theorem 1 is true.

37. To prove Theorem 2, we have to further modify our complex integral so that it becomes

$$\frac{1}{2\pi i} \int \frac{J_{m+1}(xt)\, dt}{tJ'_m(t) + HJ_m(t)} = \frac{1}{2\pi i} \int \frac{R(t)}{t}\, dt.$$

Evidently then similar reasoning applies. We can use (1) and (2) as they stand. Also, as before,

$$| R(\sigma+it) | < 2N_1 e^{tx}/e^{t} < 2N_1 e^{-t(1-x)};$$

and therefore

$$\mathfrak{R}\left[\int_0^{\tau} \frac{R(\sigma+it)}{\sigma+it}\, dt\right] \leqslant \int_0^{\tau} | R(\sigma+it) | \frac{dt}{\sqrt{(\sigma^2+t^2)}} \leqslant 2N_1 \int_0^{\tau} \sigma^{-1}\, dt$$

$$\leqslant 2N_1 \tau/\sigma_1 \leqslant 2N_1,$$

taking $\sigma = \tau$, whether $x = 1$ or not.

On the other hand

$$\int_0^{\sigma} \frac{R(s+i\tau)}{s+i\tau}\, ds \leqslant \int_0^{\sigma} \left| \frac{R(s+i\tau)}{s+i\tau} \right| ds \leqslant \int_0^{\sigma} \frac{|R(s+i\tau)|}{\sqrt{(s^2+\tau^2)}}\, ds \leqslant \frac{4N\sigma}{\tau};$$

and is therefore bounded when we put $\sigma = \tau$.

It remains to prove the other statements of Theorem 2. This may be done as follows.

To prove that, for $\eta \leqslant x \leqslant 1-\epsilon$,

$$\Sigma k^{-1} C_r J_m(k_r)\, J_{m+1}(k_r x) \to 0$$

uniformly. First, it plainly converges uniformly, as we see by substituting the asymptotic expression (viii), and so getting the sum of series of the type

$$\Sigma p^{-1} \begin{matrix}\sin \\ \cos\end{matrix} px.$$

Denote its sum by $h(x)$. Multiplying by x^{-m} and integrating term-by-term between x_1 and x_2, we get

$$\int_{x_1}^{x_2} g(x)\, x^{-m}\, dx = \left[\Sigma k_r^{-1} C_r J_m(k_r) x^{-m} J_m(k_r x)\right]_{x_1}^{x_2} = 0,$$

using (1).

Since this is true for all pairs of values x_1 and x_2 in $\eta \leqslant x \leqslant 1-\epsilon$, and $g(x)$ is continuous,

$$g(x) = 0.$$

Thus our series converges uniformly to zero.

Again, when $x = 1$, it takes the form

$$\Sigma k_r^{-1} C_r J_m(k_r)\, J_{m+1}(k_r) = 1,$$

by Theorem 3.

38. These auxiliary theorems enable us to prove quite simply the following main theorem on the behaviour of a Bessel series at $x = 1$.

THEOREM 4 (Hobson's Theorem).—*If $f(x)$ has bounded variation at $x = 1$, and $\sqrt{x}\, f(x)$ is summable*

$$\Sigma C_r J_m(k_r) \int_0^1 J_m(k_r x)\, x\, f(x)\, dx = f(1-0).$$

First we know that, in $(\eta, 1-\epsilon)$,

$$\Sigma C_r J_m(k_r)\, J_m(k_r x)$$

oscillates boundedly, and further that

$$\Sigma \int_0^x x^m C_r J_m(k_r)\, J_m(k_r x)\, dx = \Sigma x^m k_r^{-1} C_r J_m(k_r)\, J_{m+1}(k_r x)$$

converges uniformly to zero in $(\eta, 1-\epsilon)$.

Hence, by a known theorem,* since $x^{1-m}f(x)$ is summable in $(\eta, 1-\epsilon)$,

$$\Sigma \int_{\eta}^{1-\epsilon} x^m x^{1-m} C_r J_m(k_r)\, J_m(k_r x)\, dx = 0,$$

that is,

$$\Sigma\, C_r J_m(k_r) \int_{\eta}^{1-\epsilon} x f(x)\, J_m(k_r x)\, dx = 0.$$

Secondly, we know that, in $(0, \eta)$,

$$\sqrt{x}\, \Sigma\, C_r J_m(k_r)\, J_m(k_r x)$$

oscillates boundedly (Theorem 1).

Therefore by a known theorem† this series is semi-integrable above and below, and therefore, by the theorem just quoted, if $L(x)$ and $U(x)$ be the upper and lower functions of the series,

$$\int_0^{\eta} L(x)\, g(x)\, dx \leqslant \Sigma \int_0^{\eta} C_r J_m(k_r)\, J_m(k_r x)\, x^{\frac{1}{2}} g(x)\, dx \leqslant \int_0^{\eta} U(x)\, g(x)\, dx,$$

provided $g(x)$ is summable.

Hence we may write $x^{\frac{1}{2}}f(x)$ for $g(x)$, and therefore, if B be the upper bound of the values of the functions $|L(x)|$ and $|U(x)|$ in $(0, \eta)$,

$$\left| \Sigma \int_0^{\eta} C_r J_m(k_r)\, J_m(k_r x) x\, f(x)\, dx \right| \leqslant B \int_0^{\eta} x^{\frac{1}{2}} f(x)\, dx,$$

and is therefore as small as we please, if η be small enough.

Finally, we know that, in $(1-\epsilon, 1)$,

$$\Sigma\, k_r^{-1} C_r J_m(k_r)\, J_{m+1}(k_r x)$$

converges boundedly (Theorem 2). We may therefore integrate with respect to the function of bounded variation $x^{-m}f(x)$ in $1-\epsilon$ to 1, and the result, by Theorem 2, will be zero when the interval of integration does not include the point $x = 1$, and when it does include this point will be $\{f(1)-f(1-0)\}$.

---

* See my paper "Successions of Integrals and Fourier Series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 64 (1912).

† See my paper "Semi-Integrals etc.", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 306 (1910).

Now

$$\int_{1-\epsilon}^{1} J_m(k_r x) x f(x)\, dx$$

$$= \int_{1-\epsilon}^{1} x^{-m} f(x)\, d\left\{k_r^{-1} x^{1+m} J_{m+1}(k_r x)\right\}$$

$$= \left[k_r^{-1} x f(x) J_{m+1}(k_r x)\right]_{1-\epsilon}^{1} - \int_{1-\epsilon}^{1} k_r^{-1} x^{1+m} J_{m+1}(k_r x)\, d\left\{x^{-m} f(x)\right\}.$$

Multiplying both sides by $C_r J_m(k_r x)$ and summing and using the result just given, as well as Theorem 2, we get

$$\Sigma \int_{1-\epsilon}^{1} C_r J_m(k_r) J_m(k_r x) x f(x)\, dx = f(1) - \left\{f(1) - f(1-0)\right\} = f(1-0).$$

which proves the theorem.

ADDITIONAL NOTE ON TWO PROBLEMS IN THE ANALYTIC THEORY OF NUMBERS

By G. H. HARDY.

[Received November 27th, 1917.—Read January 17th, 1918.]

1. This note is a short supplement to my paper "The average order of the arithmetical functions $P(n)$ and $\Delta(n)$", published in Vol. 15 of the *Proceedings*.* I did not point out explicitly the relation of the results which I proved there to the well known, but very difficult, theorems expressed by the equations

$$P(n) = O(n^{\frac{1}{4}+\epsilon}), \qquad \Delta(n) = O(n^{\frac{1}{4}+\epsilon}). \tag{1}$$

These theorems may be deduced very simply, and by two different methods, from the theorems proved in my paper; and each deduction has points of interest. I state them in terms of $P(n)$ and the associated functions.

2. In § 2.6, I proved that†

$$\int_1^x \{U_a(\tau)\}^2 \, d\tau = O(x). \tag{2.61}$$

But it is easy to see that my argument really proves more than this, viz.,

$$\int_{x'}^x \{U_a(\tau)\}^2 \, d\tau = O(x-x'), \tag{2.61'}$$

provided only that $x-x' > K\sqrt{x}$. In fact the argument used at the top of p. 208 gives

$$\int_{x'}^x \{U_a(w^2)\}^2 \, dw = O(x-x'),$$

provided only that $x-x' > K$. If now we write $w^2 = \tau$, $x^2 = \xi$,

* Pp. 192–213.

† I retain the notation of my paper, and the numbering of formulæ quoted from it.

$x'^2 = \xi'$ (as on p. 207), and

$$W(\xi, \xi') = \int_{\xi'}^{\xi} \frac{\{U_\alpha(\tau)\}^2}{\sqrt{\tau}}\, d\tau = O(\sqrt{\xi}-\sqrt{\xi'}),$$

we have $\int_{\xi'}^{\xi} \{U_\alpha(\tau)\}^2\, d\tau = \int_{\xi'}^{\xi} \sqrt{\tau}\, \frac{d}{d\tau} W(\tau, \xi')\, d\tau$

$$< \sqrt{\xi}\, W(\xi, \xi') < (\sqrt{\xi}+\sqrt{\xi'})\, W(\xi, \xi') = O(\xi-\xi'):$$

a result equivalent to (2 . 61′), since $\sqrt{\xi}-\sqrt{\xi'} > K$ corresponds to

$$\xi-\xi' > K\sqrt{\xi}.$$

From (2 . 61′) follows, as at the beginning of § 2 . 7,

$$\frac{1}{x-x'} \int_{x'}^{x} \{S_\alpha(\tau)\}^2\, d\tau = O(x^{\frac{1}{2}+\alpha}). \tag{2 . 71′}$$

3. Suppose now that $|S_\alpha(x)| > x^\beta$,

where $0 < \beta < \frac{1}{2}+\alpha$, for a sequence of values of x, say $x_1, x_2, x_3, \ldots$, surpassing all limit. When x increases by unity, the change in $S_\alpha(x)$ is of order $O(x^{\alpha+\epsilon})$.* Hence

$$|S_\alpha(x)| > Kx^\beta$$

throughout an interval extending on either side of x_i to a distance greater than $Kx_i^{\beta-\alpha-\epsilon}$. We have therefore

$$\int_{x_i'}^{x_i} \{S_\alpha(\tau)\}^2\, d\tau > Kx_i^{3\beta-\alpha-\epsilon}, \tag{2 . 71″}$$

if $x_i-x_i' > Kx_i^{\beta-\alpha-\epsilon}$. This last inequality will certainly be satisfied if $x_i-x_i' > K\sqrt{x_i}$, since $\beta-\alpha < \frac{1}{2}$. Thus we may take $x_i-x_i' = \sqrt{x_i}$ and use (2 . 71″) in conjunction with (2 . 71′); and plainly this gives

$$x^{3\beta-\alpha-\epsilon} = O(x^{1+\alpha}),$$

or

$$\beta \leqslant \tfrac{1}{3}+\tfrac{2}{3}\alpha+\tfrac{1}{3}\epsilon.$$

We have therefore $S_\alpha(x) = O(x^{\frac{1}{3}+\frac{2}{3}\alpha+\epsilon})$,

for every pair of positive values of α and ϵ; and therefore, by the theorem quoted in the footnote to p. 200,

$$S(x) = O(x^{\frac{1}{3}+\frac{2}{3}\alpha+\epsilon}).$$

* This follows at once from the equation which immediately precedes (2 . 82), if we observe that $S_\alpha(x)$ is a continuous function of x.

Since α and ϵ are arbitrarily small, this is equivalent to the first of the equations (1).

It would seem that it should be possible to prove that

$$\frac{1}{x-x'}\int_{x'}^{x}\{\mathrm{P}(\tau)\}^2\,d\tau = O(x^{\frac{1}{2}+\epsilon}), \tag{2}$$

if only $x-x' > K\sqrt{x}$, and so to prove that the average order of $\mathrm{P}(n)$ is $O(n^{\frac{1}{4}+\epsilon})$ in a sense more precise than that in which it is proved to be so in my former paper. I have, however, not been able to complete the proof.

4. We can also deduce the first of the equations (1) from the formula

$$S_\alpha(x) = O(x^{\frac{1}{4}+\frac{1}{2}\alpha}) \quad (\alpha > \tfrac{1}{2}), \tag{2.42'}$$

an equation equivalent to (2.42) of my former paper. This deduction depends on a general arithmetic theorem proved by K. Ananda Rau.

If $a_n = O(n^\epsilon)$ *and*

$$S_\kappa(x) = \sum_{n\leqslant x}(x-n)^\kappa a_n = O(x^{\gamma+\epsilon}),$$

where $0 < \kappa < 1$ *and* $\gamma > 0$, *for every positive value of* ϵ, *then*

$$S(x) = \sum_{n\leqslant x} a_n = O\left(x^{\frac{\gamma}{1+\kappa}+\epsilon}\right)$$

for every positive value of ϵ.

This theorem, which I state in the form in which I wish to apply it, is a particular case of more general theorems proved by Ananda Rau, the aim of which is to extend, to arbitrary (non-integral) orders of integration and summation, a number of theorems proved by Mr. Littlewood and myself in a paper published in Vol. 11 of the *Proceedings*.* Dr. Marcel Riesz first succeeded in extending some of our principal theorems in this manner, and Ananda Rau has since investigated the generalisations more systematically. I shall not insert a proof here, as I hope that Ananda Rau will soon be able to publish proofs of the more general theorems.

In the present case we have

$$a_n = r_n, \qquad \kappa = \alpha > \tfrac{1}{2}, \qquad \gamma = \tfrac{1}{4}+\tfrac{1}{2}\alpha;$$

---

* "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1912, pp. 411–478.

and the conclusion is that

$$S(x) = O\left\{x^{\frac{1+2\alpha}{4(1+\alpha)}+\epsilon}\right\}.$$

Since α may be as near to $\frac{1}{2}$ as we please, and $(1+2\alpha)/4(1+\alpha) = \frac{1}{3}$ when $\alpha = \frac{1}{2}$, we have

$$S(x) = O(x^{\frac{1}{3}+\epsilon}),$$

which is equivalent to the first of the equations (1).

The *best* known "O" results concerning $P(n)$ and $\Delta(n)$ are

$$P(n) = O(n^{\frac{1}{3}}), \qquad \Delta(n) = O(n^{\frac{1}{3}} \log n).$$

These results may, as has been shown by Landau,* be deduced from the expressions of

$$\sum_{n \leqslant x} (x-n)^k r(n), \qquad \sum (x-n)^k d(n),$$

where k is a positive integer, as series of Bessel functions. But the deduction is of a much less direct character than that outlined above.

---

* "Über die Gitterpunkte in einen Kreise (II)", *Göttinger Nachrichten*, 1915: "Über Dirichlet's Teilerproblem", *Münchener Sitzungsberichte*, 1915, pp. 317–328.

ABEL'S THEOREM AND ITS CONVERSE

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Read December 6th, 1917.—Received January 17th, 1918.]

1.

INTRODUCTION AND SUMMARY.

1.1. Our main object in this paper is to obtain as far-reaching a generalisation as possible of "Tauber's Theorem", the well-known converse of Abel's famous theorem concerning power-series. It will be necessary to give a rapid summary of the results already known; and we can do this most shortly and clearly if we begin with a few definitions of a verbal character.

We shall always denote the power-series in question by $\Sigma a_n x^n$ or by S, and its sum by $f(x)$. We suppose that the radius of convergence of S is unity, and that the point on the circle of convergence which is in question is the point $x = 1$. The series Σa_n we shall call A, and we shall also use A to denote its sum, when it is convergent.

We shall use (K), (L), (O), and (o) as abbreviations for the propositions:

(K) *A is convergent,*

(L) $f(x) \to A$,

(O) $a_n = O\left(\frac{1}{n}\right)$,

(o) $a_n = o\left(\frac{1}{n}\right)$.

We shall be concerned in the sequel with certain classes of curves C along which x may approach the limit 1.* We shall call C a *path* if it is a simple Jordan curve which does not pass outside the circle: that

* We are, of course, concerned with the nature of C only in the neighbourhood of $x = 1$. It is therefore presupposed, in the definitions which follow, that their conditions need only be satisfied for values of x near enough to unity.

is to say if it is defined by equations

$$x = \xi + i\eta, \quad \xi = \phi(t), \quad \eta = \psi(t) \quad (t_0 \leqslant t \leqslant T);$$

where ϕ and ψ are continuous for $t_0 \leqslant t \leqslant T$, $\phi(T) = 1$, $\psi(T) = 0$, $\phi^2 + \psi^2 \leqslant 1$, and $\phi(t_1) = \phi(t_2)$, $\psi(t_1) = \psi(t_2)$ are not both true unless $t_1 = t_2$.

If $\phi^2 + \psi^2 < 1$ except for $t = T$, we shall call C an *internal path*. If it lies entirely between two chords of the unit circle, meeting at $x = 1$, we shall call it a *Stolz-path*.

If C possesses a continuously turning tangent at every point except $x = 1$, and approaches $x = 1$ with a definite direction, so that am $(1-x)$ tends to a limit when $x \to 1$, we shall call it a *regular path*. If the limit of am $(1-x)$ is neither $\frac{1}{2}\pi$ nor $-\frac{1}{2}\pi$, C is a *regular Stolz-path*.

Thus a chord of the unit circle, or a segment of a circle which passes through $x = 0$ and $x = 1$, and contains an angle greater than a right angle, is a regular Stolz-path. An arc of a circle which touches the unit circle internally, or an arc of the unit circle itself, is regular, but not a Stolz-path. The curve

$$\eta = (1-\xi) \sin \frac{1}{1-\xi}$$

is a Stolz-path, but not regular. The curve

$$\eta = \sqrt{(1-\xi^2)} \sin \frac{1}{\sqrt{(1-\xi^2)}}$$

is a path, but neither regular nor a Stolz-path.

1 . 2. Abel's Theorem is

A. (*K*) *implies* (*L*) *when C is the radius* (0, 1).*

Stolz's generalisation is

B. (*K*) *implies* (*L*) *when C is any Stolz-path.*†

Proofs of **A** and **B** will be found in Bromwich's *Infinite Series*.‡ To

* N. H. Abel, "Untersuchungen über die Reihe $1 + \frac{m}{1}x + \frac{m(m-1)}{1.2}x^2 + \ldots$", *Journal für Math.*, Vol. 1, 1826, pp. 311–339 (*Œuvres*, Vol. 1, pp. 219–250).

† O. Stolz, "Beweis einiger Sätze über Potenzreihen", *Zeitschrift für Math.*, Jahrgang 20, 1875, pp. 369-376; "Nachtrag ...", *ibid.*, Jahrgang 29, 1884, pp. 127–128.

See pp. 130, 210–212.

these theorems should be added

C. *It is not true that* (*K*) *implies* (*L*) *whenever C is a regular path.**

Thus, for example, the series

$$\Sigma n^{-b} e^{Ain^a} \quad (A>0,\ 0<a<1) \tag{1.21}$$

is convergent whenever $b>1-a$; but the associated power-series $f(x)$ does not tend to a limit if $b<1-\frac{1}{2}a$ and C is an arc of a circle touching the unit circle.

Tauber's Theorem is

D. (*L*) *and* (*o*) *imply* (*K*) *when C is the radius* (0, 1).†

This theorem has been generalised in several directions. The generalisations with which we shall be most directly concerned are

E. (*L*) *and* (*o*) *imply* (*K*) *when C is any Stolz-path.*‡

F. (*L*) *and* (*O*) *imply* (*K*) *when C is the radius* (0, 1).§

But we must also mention

G. *In either* **D** *or* **E**, (*o*) *may be replaced by the more general condition*

$$a_1+2a_2+\ldots+na_n=o(n).$$

This condition is also necessary for the truth of (*K*).‖

H. *In* **F**, (*O*) *may be replaced by the condition that* a_n *is real and* na_n *bounded above or below.*¶

---

* G. H. Hardy and J. E. Littlewood, **1** (see the list of papers in 1.5), p. 475 (Theorem 47). The proof is not given, but the materials necessary for one will be found in a paper by Hardy, "A theorem concerning Taylor's series", *Quarterly Journal*, Vol. 44, 1913, pp. 147–160 (pp. 150 *et seq.*).

† A. Tauber, "Ein Satz aus der Theorie der unendlichen Reihen", *Monatshefte für Math.*, Vol. 8, 1897, pp. 273–277. See also Bromwich, *Infinite Series*, p. 251, or Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1916, p. 40.

‡ E. Landau, "Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande Konvergenzgebietes", *Monatshefte für Math.*, Vol. 18, 1907, pp. 8–28. See also Bromwich and Landau, *l.c. supra.*

§ J. E. Littlewood, "The converse of Abel's theorem on power-series", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, 1911, 434–448.

‖ Tauber (*l.c. supra*) proves this when C is the radius. We cannot refer to an explicit proof for the case in which C is an arbitrary Stolz-path; but the result is an immediate consequence of the arguments used by Tauber and by Landau.

¶ That is to say, $na_n<H$ or $na_n>-H$, where H is a constant. G. H. Hardy and J. E. Littlewood, **2**; see also Landau's book referred to above, pp. 45–56. The condition is plainly more general than (*O*) when a_n is real.

These theorems can only be appreciated if we bear in mind other results of a negative character. The trivial example

$$1-1+1-\ldots,$$

is enough to show that the existence of Abel's limit does not involve the convergence of the series; thus "(L) implies (K)", the straightforward converse of Abel's Theorem, is false. It is not quite so easy to find a similar example in which the terms of the series tend to zero. This was first done by Pringsheim*; but a more natural example is provided by the series (1.21) when $0 < b < 1-a$. Here $a_n = O(n^{-b})$, and a may be as small, and so b as nearly equal to 1, as we please. Thus no condition of this type, with $b < 1$, is sufficient to ensure the convergence of the series whenever Abel's limit exist. This suggests that **F** is really a "best possible" theorem of its kind; and this is shown by the theorem

K. *There is no function $\phi(n)$, such that $\phi(n) \to \infty$ and*

$$a_n = O\left\{\frac{\phi(n)}{n}\right\},$$

together with (L), implies (K).†

1.3. No extension of **F** to paths other than the radius has yet been published. The extension of theorems of the "o" character to paths other than Stolz-paths was first considered seriously in our paper **1.**‡ In this paper we confined ourselves to *regular* paths, and we found that, in order to obtain satisfactory results, it was essential to replace (L) by a different condition. This condition is

$$(\Lambda) \qquad \Phi(x) = \frac{1}{1-x}\Sigma\frac{a_n}{n+1}(1-x^{n+1}) \to A.$$

It is to be observed that, so long as C is regular, and

$$\Sigma\frac{a_n}{n+1}$$

is absolutely convergent, we have

$$(1.31) \qquad \Phi(x) = \frac{1}{1-x}\int_x^1 f(t)\,dt,$$

* A. Pringsheim, "Über die Divergenz gewisser Potenzreihen an der Konvergenzgrenze", *Münchener Sitzungsberichte*, Vol. 31, 1901, pp. 505-524.

† J. E. Littlewood, *l.c.*, p. 444 (Theorem C)..

‡ Pp. 475-477.

the integration being effected along C. It is an easy deduction that $\Phi(x) \to A$ in all cases in which $f(x) \to A$, whereas the converse is untrue. Thus (L) implies (Λ), and (Λ) is a generalisation of (L), at any rate in all such cases as we were considering before. This being so, we proved

L. (Λ) *and* (o) *imply* (K) *whenever* C *is a regular path.*

As a corollary we have

M. (L) *and* (o) *imply* (K) *whenever* C *is a regular path.*

This theorem includes **D** and **E** as special cases. We also proved* the direct converse of **L**, viz.

N. (K) *and* (o) *imply* (Λ) *whenever* C *is a regular path.*

This theorem becomes untrue if (Λ) is replaced by (L). It is an "Abelian" theorem, but differs fundamentally from the ordinary Abelian theorems **A** and **B** in that its truth depends upon a condition such as occurs in the "Tauberian" theorems. It is also unlike all the theorems which precede in being reversible: and, on combining it with **L**, we obtain

O. *If* A *satisfies* (o), *then the necessary and sufficient condition for its convergence is that* (Λ) *should be true when* x *tends to* 1, *either along any regular path, or along all.*

1.4. We begin our new investigations by a direct extension of **F** to a regular Stolz-path, viz.

P. (L) *and* (O) *imply* (K) *when* C *is a regular Stolz-path.*

This theorem is included in others which come after and are proved in a quite different way. But the method we use (in 2.1) seems to us of considerable interest in itself.

In 2.2 and the succeeding paragraphs we attack our main problem. Our object is to generalise **L**, (i) by replacing (o) by (O), and (ii) by getting rid of the restriction that C is a regular path; and the result is

Q. (Λ) *and* (O) *imply* (K), *for any path* C.

When C is regular, we can deduce as a corollary

R. (L) *and* (O) *imply* (K) *whenever* C *is a regular path.*

* The proof of this theorem (Theorem 50) is not stated explicitly, but is virtually contained in that of the preceding Theorem 49 (**L** of this paper).

We can also prove

S. *(L) and (O) imply (K) whenever C is a Stolz-path.*

But we cannot here get rid of *all* restrictions upon C.

In 3 we proceed to the corresponding Abelian theorems, and generalise **N** by proving

T. *(K) and (O) imply* (Λ), *for any path C.*

And by combining **Q** and **T**, we obtain

U. *If A satisfies (O), then the necessary and sufficient condition for its convergence is that* (Λ) *should be true when x tends to* 1, *either along any path C, or along all.*

This theorem affords the complete generalisation of **O** in each of the desired directions, and is far more comprehensive than any known theorem of its kind.

There are but few questions which remain to be answered. There is one to which we have already alluded and are unable to answer, namely whether **R** (or **S**) is true without any restriction on C. The others are connected with the question whether **T** is the "best possible" theorem of its kind. We prove

V. *(K) and (O) do not necessarily imply (L) for all paths C*:

so that the (Λ) of **T** certainly cannot be replaced by (L). And almost the same example suffices to prove

W. *(K) does not necessarily imply* (Λ) *for all paths C.*

Thus T certainly becomes untrue if the condition (O) is simply omitted. But it is desirable to prove more, viz. that (O) cannot be replaced by any less restrictive condition of the type which occurs in **K**, and we have not yet succeeded in establishing this by means of an example. If we could do this, and also remove the restriction on C in **R** (or **S**), we could fairly claim that our problem had been completely solved.

1.5. If, in Theorems **Q**–**U**, we suppose that C is an arc of the unit circle itself, we obtain a number of theorems concerning the convergence of a series

$$\Sigma(a_n+i\beta_n)\,e^{ni\theta} = \Sigma(a_n\cos n\theta+\beta_n\sin n\theta)+i\Sigma\,(a_n\sin n\theta-\beta_n\cos n\theta),$$

where $$a_n = O\left(\frac{1}{n}\right), \quad \beta_n = O\left(\frac{1}{n}\right);$$

theorems, that is to say, concerning the simultaneous convergence of a Fourier series and its *conjugate* or *allied* series. It is, however, more natural and more interesting to consider the two series independently. We prove first*

X. *If a_n and b_n satisfy (O), so that*

$$\Sigma(a_n \cos n\theta + b_n \sin n\theta)$$

is certainly the Fourier series of a summable function $f(\theta)$, then the necessary and sufficient condition that the series should converge to the sum A is that

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A$$

when $\alpha \to 0$.

When (O) is replaced by (o), Theorem **X** reduces to a theorem of Fatou.† These theorems correspond to **O** and **U**. The remaining theorems of the paper are of a somewhat different character: the most interesting of them are **Y** and **Z**, which are concerned with the Fourier series of bounded functions, and do not depend upon conditions such as (o) or (O).

We conclude these introductory remarks by giving a list of the papers of our own to which we shall have to refer. They are:—

1. "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1913, pp. 411–477.

2. "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *ibid.*, Vol. 13, 1914, pp. 174–191.

3. "Some theorems concerning Dirichlet's series", *Messenger of Math.*, Vol. 43, 1914, pp. 134–147.

4. "Theorems concerning the summability of series by Borel's exponential method", *Rendiconti del Circ. Mat. di Palermo*, Vol. 41, 1916, pp. 36–53.

5. "Sur la convergence des séries de Fourier et des séries de Taylor", *Comptes Rendus*, December 24, 1917.

* We revert to the notation usual in the theory of Fourier series.

† P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Mathematica*, Vol. 30, 1906, pp. 335–400 (p. 385).

2.

THE TAUBERIAN THEOREMS.

Proof of Theorem **P**.

2.1. THEOREM **P**.—*If*

(*O*) $$a_n = O\left(\frac{1}{n}\right),$$

and (*L*) $$f(x) = \Sigma a_n x^n \to A,$$

when $x \to 1$ *along a regular Stolz-path C, then* Σa_n *converges to the sum* A.

We suppose, as obviously we may without loss of generality, that $|na_n| < 1$ and $A = 0$. We write

$$x = re^{i\theta} = e^{-\rho+i\theta},$$

and we suppose first that C is the particular curve

(2.11) $$\theta = k\rho.$$

Then, if $$F_k(\rho) = \Sigma a_n e^{-(1-ki)n\rho},$$

we have $$F_k(\rho) = o(1)$$

when $\rho \to 0$, and

$$\rho^p F_k^{(p)}(\rho) = (-1)^p (1-ki)^p \rho^p \Sigma n^p a_n e^{-(1-ki)n\rho}$$
$$= O(\rho^p \Sigma n^{p-1} e^{-n\rho}) = O(1),$$

for every positive integral value of p. It follows, from our fundamental theorems on derivatives*, that

(2.12) $$\rho^p F_k^{(p)}(\rho) = o(1),$$

for every such value of p.

We shall now prove that

(2.13) $$F_l(\rho) = o(1),$$

for any value of l such that $k-1 \leqslant l \leqslant k+1$.

---

* The particular theorem required is obtained by supposing $\phi = \psi = 1$ in Case (*b*) of Theorems 6 and 8 of our paper **1**.

If $l = k+\delta$, so that $|\delta| \leqslant 1$, we have

$$F_l(\rho) = \Sigma\, a_n e^{-(1-ki)\,n\rho}\, e^{\delta i n\rho}$$

$$= \sum_{(n)} a_n e^{-(1-ki)\,n\rho} \left\{ \sum_{p=0}^{P-1} \frac{(\delta i n\rho)^p}{p!} + \Delta_P \right\}$$

$$= \sum_{p=0}^{P-1} \frac{(\delta\rho)^p}{p!} \sum_{(n)} n^p a_n e^{-(1-ki)\,n\rho} + R_P,$$

where

$$|\Delta_P| < \frac{2(\delta n\rho)^P}{P!},$$

$$|R_P| < \frac{2(\delta\rho)^P}{P!} \Sigma n^P |a_n| e^{-n\rho}.$$

Now $|na_n| < 1$, and

$$\Sigma n^{P-1} e^{-n\rho} < \int_0^\infty \xi^{P-1} e^{-\xi\rho} d\xi + 2\,\mathrm{Max}\,(\xi^{P-1} e^{-\xi\rho})$$

$$= \frac{\Gamma(P)}{\rho^P} + 2\left(\frac{P-1}{\rho}\right)^{P-1} e^{-(P-1)}$$

$$< \frac{2\Gamma(P)}{\rho^P},$$

for $0 < \rho < \rho_0$, where ρ_0 is a number independent of P. Hence

$$|R_P| < \frac{4\delta^P}{P} \leqslant \frac{4}{P}. \tag{2.14}$$

And therefore

$$F_l(\rho) = \sum_{(p)} \frac{(\delta i\rho)^p}{p!} \sum_{(n)} n^p a_n e^{-(1-ki)\,n\rho}$$

$$= \sum_{(p)} \frac{1}{p!} \left(-\frac{\delta i}{1-ki}\right)^p \rho^p F_k^{(p)}(\rho);$$

and this series is, in virtue of (2.14), *uniformly convergent for* $|\delta| \leqslant 1$ *and* $\rho > 0$. But, by (2.12), every term of the series tends to zero. Hence $F_l(\rho)$ tends to zero*: that is to say, we have proved (2.13).

It follows, by the repeated application of this argument, that if a_n satisfies (O) and $f(x)$ tends to zero along any Stolz-path of the type (2.11),

---

* Our argument here is the same in principle as that which we used in the proof of the "general Borel-Tauber" theorem: see our paper **4**, p. 44.

then it tends to zero along any other Stolz-path of the same type. In particular it tends to zero along the real axis, to which the path reduces when $k = 0$. And hence, by Theorem **F**, the series Σa_n converges to zero.

The theorem is thus proved for paths of the special type (2 . 11). In general, the equation of a regular Stolz-path may be written in the form

$$\theta = k\rho + o(\rho). \tag{2 . 15}$$

It is easy to see that, if $f(x)$ tends to zero along (2 . 15), it also tends to zero along (2 . 11). For, if (ρ, θ) and (ρ, θ') are corresponding points on the paths (2 . 11) and (2 . 15), we have

$$| e^{ni\theta} - e^{ni\theta'} | = | e^{ni(\theta-\theta')} - 1 | = o(n\rho)$$

when $\rho \to 0$, and so

$$f(e^{-\rho+i\theta}) - f(e^{-\rho+i\theta'}) = o(\rho)\Sigma n | a_n | e^{-n\rho} = o(1).$$

The truth of the theorem in its general form now follows from the argument which precedes.

Proof of Theorem **Q**.

2 . 2. THEOREM **Q**.—*If*

$$a_n = O\left(\frac{1}{n}\right), \tag{O}$$

and

$$\Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1} (1 - x^{n+1}) \to A, \tag{A}$$

when $x \to 1$ *along some path* C, *then* Σa_n *converges to the sum* A.

We may suppose that $A = 0$, $a_0 = 0$, and $|na_n| < 1$. And we shall begin by proving

LEMMA a.—(O) *being satisfied, the necessary and sufficient condition that*

$$s_n = a_1 + a_2 + \ldots + a_n$$

should be bounded is that $\Phi(x)$ *should be bounded for* $|x| \leqslant 1$, $x \neq 1$; *and this condition is satisfied if* $\Phi(x)$ *is bounded when* $x \to 1$ *along any particular path* C.

We take $$\nu = \left[\frac{1}{|1-x|}\right],$$

and we may suppose $|1-x| < 1$, so that

$$\tfrac{1}{2} < \nu|1-x| \leqslant 1.*$$

Then

$$\Phi(x) = \left(\sum_0^{\nu-1} + \sum_\nu^\infty\right) \frac{a_n}{n+1} \frac{1-x^{n+1}}{1-x} = \phi_1 + \phi_2, \tag{2.21}$$

say. In the first place we have

$$|\phi_2| < \frac{2}{|1-x|} \sum_\nu^\infty \frac{1}{n(n+1)} = \frac{2}{\nu|1-x|} < 4. \tag{2.22}$$

Secondly,

$$1 - \frac{1-x^{n+1}}{(n+1)(1-x)} = \frac{1}{n+1}\{(1-x)+(1-x^2)+\ldots+(1-x^n)\},$$

$$\left|1 - \frac{1-x^{n+1}}{(n+1)(1-x)}\right| < \frac{|1-x|}{n+1}(1+2+\ldots+n) = \tfrac{1}{2}n|1-x|,$$

$$|s_{\nu-1} - \phi_2| \leqslant \sum_1^{\nu-1} |a_n| \left|1 - \frac{1-x^{n+1}}{(n+1)(1-x)}\right| < \tfrac{1}{2}|1-x| \sum_1^{\nu-1} |na_n| \tag{2.23}$$

$$< \nu|1-x| \leqslant 1.$$

From (2.21), (2.22), and (2.23), we obtain

$$|s_{\nu-1} - \Phi(x)| < 5;$$

which proves both parts of the lemma, since ν passes through an unbroken sequence of integral values when $x \to 1$ along C.

2.3. The remainder of the proof of Theorem **Q** depends upon certain lemmas in the theory of analytic functions, of the general type associated particularly with the names of Phragmén and Lindelöf.†

---

* It is easily verified that $y[1/y] > \frac{1}{2}$ if $0 < y < 1$.

† See in particular the memoir "Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier", *Acta Mathematica*, Vol. 31, 1908, pp. 381–406. The chief results of this memoir may now be regarded as classical.

LEMMA β.*—*Suppose that T is the semi-infinite strip, in the plane of the complex variable $w = u+iv$, defined by the inequalities $0 \leqslant u \leqslant \pi$, $v \geqslant v_0$; that* **A** *and* **B** *are the left and right-hand edges of the strip, and that* **C** *is a simple Jordan curve which lies entirely inside T, extends to infinity, and divides T into two regions L and R. Suppose further that $f(w)$ is regular inside T and continuous and bounded throughout T.*

Finally, suppose that

$$\overline{\lim}\ |f(w)| \leqslant h, \tag{2.31}$$

where $h > 0$, when w tends to infinity along **A** *(or* **B***) and along* **C**. *Then* (2.31) *holds when w tends to infinity in any manner inside L (or R).*

Let us suppose that the data refer to **A** and **C**. We can choose $v_1(\epsilon)$ so that

$$|f| < h+\epsilon, \tag{2.33}$$

at all points of **A** and **C** for which $v > v_1$. We can then draw a cross-cut (*Querschnitt*) Q, cutting off from L an infinite region L_Q; and (2.33) will be satisfied at all points of $\mathbf{A}_Q$ and $\mathbf{C}_Q$, the parts of **A** and **C** which belong to the boundary of L_Q.

Let K be the upper bound of $|f|$ on Q. We can now choose

$$p(\epsilon, v_1, K) = p(\epsilon),$$

so that

$$\left|\frac{w}{w+p}\right| < 1,$$

---

* We state this and the following lemmas in the special forms in which they are required for our immediate purpose. All of them, naturally, are capable of wide generalisation. The most interesting of these generalisations is the following:

If (1) *T is the strip $\alpha \leqslant u \leqslant \beta$, $v \geqslant v_0$;* **A** *and* **B** *its edges;* $\mathbf{C}_\alpha$ *and* $\mathbf{C}_\beta$ *simple non-intersecting Jordan curves interior to T and asymptotic to* **A** *and* **B**; *and* **C** *a similar curve asymptotic to $u = \nu$, where $\alpha < \nu < \beta$:*

(2) *$f(w)$ is regular inside and continuous throughout the region T' formed by those points of T which lie to the right of* $\mathbf{C}_\alpha$ *and to the left of* $\mathbf{C}_\beta$:

(3) $\overline{\lim}\ |f(w)| \leqslant \mathbf{a}$ *when w tends to infinity along* $\mathbf{C}_\alpha$, *and* $\overline{\lim}\ |f(w)| \leqslant \mathbf{b}$ *when w tends to infinity along* $\mathbf{C}_\beta$:

$$f(w) = O(e^{e^{cv}}), \tag{4}$$

where $c < \pi/(\beta-\alpha)$, uniformly throughout T': then

$$\lim\ |f(w)| \leqslant \mathbf{a}^{(\beta-\nu)/(\beta-\alpha)}\,\mathbf{b}^{(\nu-\alpha)/(\beta-\alpha)},$$

when w tends to infinity along **C**. *This result still holds when* **a** *or* **b** *or both are zero.*

on $\mathbf{A}_Q$ and $\mathbf{B}_Q$, and $$\left| \frac{w}{w+p} \right| < \frac{h+\epsilon}{K}$$

on Q. And if we write $$F = \frac{wf}{w+p},$$

we have $$|F| < h+\epsilon$$

at all points on the boundary of L_Q; and therefore* at all points of L_Q.

Hence $$\overline{\lim} |F| \leqslant h,$$

when w tends to infinity in any manner inside L; and therefore

$$\overline{\lim} |f| \leqslant h,$$

which proves the lemma.

LEMMA γ.—*Suppose that* (2.31) *is satisfied on both* **A** *and* **B**, *and that*

(2.34) $$\overline{\lim} |f(w)| \leqslant \delta,$$

where $0 < \delta < h$, *on* **C**. *Then*

(2.35) $$\overline{\lim} |f(w)| \leqslant \sqrt{(\delta h)},$$

when w *tends to infinity along* **M**, *the straight line equidistant between* **A** *and* **B**.

We denote by M_L and M_R those parts of M which lie in L and R respectively.

Write $$g = e^{kw} f,$$

where $$e^{k\pi} = h/\delta.$$

Then $$\overline{\lim} |g| \leqslant h,$$

on A, and $$\overline{\lim} |g| \leqslant e^{k\pi}\delta = h,$$

on **C**. Hence $$\overline{\lim} |g| \leqslant h,$$

on M_L; and so $$\overline{\lim} |f| \leqslant he^{-\frac{1}{2}k\pi} = \sqrt{(\delta h)},$$

on M_L. Similarly, using an auxiliary function

$$g = e^{k(\pi - w)} f,$$

we can show that (2.35) holds on M_R, and so on the whole of M.

LEMMA δ.—*If* $f(w) = O(1)$ *throughout* T, *and* $f(w) = o(1)$ *on* **C**, *then* $f(w) = o(1)$ *on* M.

---

* Phragmén and Lindelöf, *l.c.*, p. 388.

Write $$g = f + \delta,$$

where $0 < \delta < 1$. Then we can choose an h independent of δ and such that the conditions of Lemma γ are satisfied; and

$$\overline{\lim}\, |g| \leqslant \sqrt{(\delta h)}$$

on M. Hence $$\overline{\lim}\, |f| \leqslant \sqrt{(\delta h)} + \delta$$

on M; which proves the theorem, since δ is arbitrarily small.

2.4. We now transform Lemma δ, by means of the theory of conformal representation, into a proposition suitable for direct application to the theory of power-series.

The equation $$\frac{x-1}{x+1} = ie^{iw}$$

transforms the strip $0 \leqslant v \leqslant \pi$ into the unit circle in the plane of x. The point $w = \frac{1}{2}\pi$ corresponds to $x = 0$, the lines **A** and **B** to the upper and lower halves of the circle, and the line M to the real diameter. The upper and lower ends of the strip correspond to $x = 1$ and $x = -1$ respectivbly. If, finally, we observe that

$$\left| \frac{1+x}{1-x} \right| = e^{v},$$

we obtain

LEMMA ϵ.—*Suppose that $\Phi(x)$ is regular for $|x| < 1$, and continuous and bounded for $|x| \leqslant 1$, $x \neq 1$. Suppose further that $\Phi(x) = o(1)$ when x tends to 1 along a certain internal path C. Then $\Phi(x) = o(1)$ when x tends to 1 by real values.*

2.5. We can now prove our main theorem. In the first place $\Phi(x)$ is continuous at all points in or on the circle, except perhaps the point $x = 1$.* We may therefore suppose, without loss of generality, that C is an internal path.†

---

* The series $$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = \Sigma\, O\left(\frac{1}{n^2}\right)$$

is plainly uniformly convergent.

† We can replace C by an internal path C', which differs so little from C that $\Phi(x)$ tends to zero along C'.

Since Φ tends to a limit along C, s_n is bounded, by Lemma a; and therefore, by the same lemma, Φ is bounded for $|x| \leqslant 1$, $x \neq 1$. Hence Φ satisfies all the conditions of Lemma ϵ; and so

$$\Phi(x) = o(1) \tag{2.51}$$

when x tends to 1 along the real axis.

Suppose then that x is real. We have

$$\Phi'(x) = \frac{1}{1-x}\Phi(x) - \frac{1}{1-x}f(x),$$

$$\Phi''(x) = \frac{2}{(1-x)^2}\Phi(x) - \frac{2}{(1-x)^2}f(x) - \frac{1}{1-x}f'(x).$$

Now $\Phi(x)$ is bounded; $f(x)$ is bounded, since s_n is bounded; and

$$f'(x) = \Sigma n a_n x^{n-1} = \Sigma O(1)\, x^n = O\left(\frac{1}{1-x}\right).$$

Hence

$$\Phi''(x) = O\left\{\frac{1}{(1-x)^2}\right\}. \tag{2.52}$$

From (2.51) and (2.52), we deduce, by Theorem 8 of our paper **1**,

$$\Phi'(x) = o\left(\frac{1}{1-x}\right);$$

and so
$$f(x) = \Phi(x) - (1-x)\Phi'(x) = o(1).$$

That is to say, *Abel's limit for $f(x)$ exists when $x \to 1$ by real values;* and therefore, by Theorem **F**, the series Σa_n is convergent.

Theorems **R** *and* **S**.

2.6. When C is *regular*,

$$f(x) \to A \tag{L}$$

implies

$$\Phi(x) \to A. \tag{A}$$

We have therefore

THEOREM **R**.—*If (O) is satisfied, and $f(x) \to A$ when $x \to 1$ along some regular path C, then Σa_n converges to the sum A.*

This theorem of course includes **P** as a special case. It is in the most

essential respects as complete a generalisation of **F** as could be desired, for, although it involves a considerable limitation as to the nature of C, there is no limitation at all on the contact of C with the circle. The contact may be as close as we please, or C may be the circle itself: and it is questions of contact rather than of regularity which are of the first interest in these investigations.

It is, however, natural to suppose that *no* limitation on C is necessary; and, if this be so, it is very desirable that it should be proved. It would seem, however, that some important change in our argument would be needed for such a proof; for, unless C is subject to considerable restrictions, it is not possible to deduce the behaviour of Φ from that of f by means of its representation as an integral taken along C.

It is interesting to observe that, if we limit C to be a Stolz-path, we can get rid of the restriction that it is to be regular. We shall only sketch the proof, the general lines of which are as follows. We draw two chords of the circle through $x = 1$, including C between them, and we consider the region T formed by points which lie between these chords and within a certain distance of $x = 1$.

By a slight modification of the proof of Theorem **E** given by Landau, we can prove that s_n is bounded, and so that $f(x)$ is bounded in T.* And by an adaptation of the arguments used in §§ 2.3–4, we can show that f tends to a limit when $x \to 1$ along any regular Stolz-path inside T. If the real axis satisfies this condition, we can appeal to Theorem **F**; if not, to Theorem **P** or **R**. In any case, we obtain

THEOREM **S**.—*If* (O) *is satisfied and* $f(x) \to A$ *when* $x \to 1$ *along some Stolz-path* C, *then* Σa_n *converges to the sum* A.

3.

THE ABELIAN THEOREMS.

Proof of Theorem **T**.

3.1. THEOREM **T**.—*If*

$$(O) \qquad a_n = O\left(\frac{1}{n}\right)$$

and

$$(K) \qquad \Sigma a_n = A,$$

* The argument fails at this stage if C is not a Stolz-path. We could not prove that f is bounded in T if the boundaries of T touched the circle.

then

$$(\Lambda) \qquad \Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1}(1-x^{n+1}) \to A,$$

when $x \to 1$ *in any manner.*

One observation should be made before we proceed to the proof. The enunciation contains no reference to a *path*: we may say, of course, "when $x \to 1$ along any path C"; but the idea of approach along a continuous path is not really relevant. In this respect there is an essential difference between this theorem and the Tauberian theorems.

We may suppose, as before, that $A = 0$ and $|na_n| < 1$. That $\Phi(x)$ is *bounded* follows from Lemma a; but to prove convergence to a limit requires an argument of somewhat greater subtlety.

What we have to prove is that

$$(3.11) \qquad S(x) = \Sigma \frac{a_n}{n+1}(1-x^{n+1}) = o(1-x).$$

We write

$$(3.12) \qquad S = \sum_0^{m-1} + \sum_m^{\infty} = S_1 + S_2,$$

where

$$(3.13) \qquad m = K\nu = K\left[\frac{1}{|1-x|}\right],$$

K being a (large) parameter. It is plain that we may suppose $|1-x|$ small enough to ensure that

$$(3.14) \qquad \frac{K}{2|1-x|} < m \leqslant \frac{K}{|1-x|}.$$

As regards S_2, we have

$$(3.15) \qquad |S_2| < 2\sum_m^{\infty} \frac{1}{n(n+1)} = \frac{2}{m} < \frac{4}{K}|1-x|.$$

In order to deal with S_1, we require a lemma.

3.2. LEMMA ξ.—*If* Σa_n *is convergent, and*

$$t_n = \sum_n^{\infty} \frac{a_\mu}{\mu+1},$$

then (i) $$t_n = o\left(\frac{1}{n}\right),$$

and (ii) Σt_n *converges to the same sum as* Σa_n.

The proof of this lemma is very simple. We suppose, as before, that $A = 0$. We can choose n_0 so that

$$|a_n+a_{n+1}+\ldots+a_\mu| < \delta,$$

for $n_0 \leqslant n < \mu$, and then

$$(3.21) \qquad |t_n| = \left| \sum_n^\infty \frac{a_n+a_{n+1}+\ldots+a_\mu}{(\mu+1)(\mu+2)} \right| < \frac{\delta}{n+1}.$$

The same argument shows that

$$(3.22) \qquad |t_n| < \frac{\mathbf{m}}{n+1},$$

for all values of n, $\mathbf{m}$ being the maximum of $|a_n+a_{n+1}+\ldots+a_\mu|$ for all values of n and μ.

Finally, we have

$$\sum_0^n t_\mu = \sum_0^n a_\mu + (n+1) \sum_{n+1}^\infty \frac{a_\mu}{\mu+1},$$

and the last term tends to zero. We have thus

$$(3.23) \qquad \left| \sum_0^n t_\mu \right| < \delta,$$

for $n \geqslant n_1$.

3.3. Now

$$(3.31) \qquad S_1 = \sum_0^{m-1} \frac{a_n}{n+1}(1-x^{n+1}) = \sum_0^{m-1} (t_n-t_{n+1})(1-x^{n+1})$$

$$= (1-x) \sum_0^{m-1} t_n - (1-x) \sum_0^{m-1} t_n(1-x^n) - t_m(1-x^m)$$

$$= S_1' + S_1'' + S_1''',$$

say. Plainly

$$(3.32) \qquad |S_1'''| < \frac{2\delta}{m} < \frac{4\delta}{K}|1-x|,$$

if $m \geqslant n_0$, which is certainly so when $|1-x|$ is sufficiently small. Secondly

$$(3.33) \qquad |S_1'| < \delta|1-x|,$$

by (3.23), if $m \geqslant n_1$, which is certainly so when $|1-x|$ is sufficiently small.

Finally,

$$(3.34)\quad |S_1''| < |1-x|^2 \sum_0^{m-1} n|t_n| = |1-x|^2 \sum_0^{n_0-1} n|t_n| + |1-x|^2 \sum_{n_0}^{m-1} n|t_n|$$
$$< \mathbf{m} n_0 |1-x|^2 + \delta m |1-x|^2$$
$$< \mathbf{m} n_0 |1-x|^2 + 2K\delta |1-x|.$$

From (3.12), (3.15), and (3.31)–(3.34), it follows that

$$(3.35)\quad |S| < |1-x| \left(\frac{4}{K} + 2K\delta + \delta + \frac{4\delta}{K} + \mathbf{m} n_0 |1-x|\right).$$

We can choose $K = K(\epsilon)$ so that $4/K < \frac{1}{3}\epsilon$; then $\delta = \delta(\epsilon, K) = \delta(\epsilon)$ so that

$$2K\delta + \delta + \frac{4\delta}{K} < \tfrac{1}{3}\epsilon;$$

and then $\eta = \eta(\epsilon, K, \delta) = \eta(\epsilon)$ so that (3.35) is satisfied, and

$$\mathbf{m} n_0 |1-x| < \tfrac{1}{3}\epsilon$$

if $|1-x| < \eta$. Thus $\quad |S| < \epsilon |1-x|$

if $|1-x| < \eta$; which proves the theorem.

Combining Theorems **Q** and **T**, we obtain

THEOREM **U**.—*If the coefficients of the series* Σa_n *satisfy* (*O*), *then the necessary and sufficient condition for its convergence is that* $\Phi(x)$ *should tend to a limit, either along any particular path C, or along all.*

Proof of Theorems **V** *and* **W**.

3.4. Theorem **T** leaves an important question unanswered. Is it certain that the hypotheses do not imply, what is more* than the theorem asserts, that

$$(L)\qquad f(x) \to A$$

along C?

Theorem **B** shows that this is so when C is a Stolz-path: in this case

* (*L*) asserts more than (A) at any rate when *C* is regular, that is to say in all ordinary cases (cf. pp. 219–220).

indeed the convergence of the series ensures the existence of the limit without any hypothesis as to the order of a_n. Theorem **C** shows that this at any rate ceases to be true when C is allowed to touch the circle. In the example which we attached to our statement of Theorem **C**, we have to take

$$1-a < b \leqslant 1-\tfrac{1}{2}a\,;$$

and these inequalities allow b to exceed any number less than 1. It follows* that no condition of the type $a_n = O(n^{-b})$, where $b<1$, is enough, in conjunction with the convergence of the series, to ensure the existence of Abel's limit along all tangential paths.

We shall now show that even the condition

$$(o) \qquad a_n = o\left(\frac{1}{n}\right)$$

is not enough for this purpose. It would be easy to modify our argument in such a manner as to show that *no* condition of the form

$$a_n = O(\chi_n),$$

where χ_n is a steadily decreasing function such that $\Sigma\chi_n$ is divergent, is enough: for simplicity, however, we take

$$\chi_n = \frac{1}{n\log n}.$$

THEOREM **V.**—*It is possible to find a convergent series Σa_n for which*

$$a_n = O\left(\frac{1}{n\log n}\right),$$

and a regular path C, such that $f(x)$ does not tend to a limit when x tends to 1 along C.

We take
$$a_n = \frac{1}{n\log n}\sin\frac{n\pi}{j},$$

if
$$n_j = e^{e^{e^j}} < n < n_{j+1} = e^{e^{e^{j+1}}}.$$

To prove that Σa_n is convergent, we write

$$w_{j,k} = \sum_{n_j < n \leqslant k < n_{j+1}} a_n, \qquad w_j = \sum_{n_j < n < n_{j+1}} a_n,$$

* Compare p. 208, where we used the same series in a different manner to establish the same point about the Tauberian Theorem **F**

so that $$w_j = w_{j,\,[n_{j+1}]}.$$

Since $n \log n$ increases steadily with n, and $\Sigma \sin(n\pi/j)$, taken between any limits, is not greater in absolute value than $2 \operatorname{cosec}(\pi/j)$, we have

$$w_{j,\,k} = O\left(\frac{j}{n_j \log n_j}\right), \qquad w_j = O\left(\frac{j}{n_j \log n_j}\right).$$

Since $$\Sigma \frac{j}{n_j \log n_j}$$

is convergent, Σw_j is absolutely convergent; and since $w_{j,k} = o(1)$, Σa_n is convergent.

Now let $$f(x) = \Sigma a_n x^n,$$

$$\phi(r, \theta) = \mathrm{I}[f(re^{i\theta})] = \Sigma a_n r^n \sin n\theta.$$

We shall show that $\phi(r, \theta)$ does not tend to a limit when $x \to 1$ along a regular path which has sufficiently close contact with the upper half of the unit circle.

To prove this we observe first that the series

(3.41) $$\Sigma a_n \sin n\theta$$

is convergent when θ is positive. In fact, if we write

$$u_{j,\,k} = \sum_{n_j < n \leqslant k < n_{j+1}} a_n \sin n\theta, \qquad u_j = \sum_{n_j < n < n_{j+1}} a_n \sin n\theta,$$

and j is so large that $\pi/j < \frac{1}{2}\theta$, we find, by the same argument that was used above, that

$$u_{j,\,k} = \tfrac{1}{2} \Sigma \frac{1}{n \log n} \left\{ \cos n\left(\theta - \frac{\pi}{j}\right) - \cos n\left(\theta + \frac{\pi}{j}\right) \right\}$$

$$= O\left(\frac{1}{\theta n_j \log n_j}\right),$$

and in particular that $$u_j = O\left(\frac{1}{\theta n_j \log n_j}\right).$$

From these relations the convergence of (3.41) follows immediately.

If now we denote the sum of (3 . 41) by $\psi(\theta)$, we have

$$\psi(\theta) = \lim_{r\to 1} \phi(r, \theta),$$

for every positive θ. It is thus sufficient to show that $\psi(\theta)$ does not tend to a limit when $\theta \to 0$. Now

$$(3\,.\,42)\quad \psi\left(\frac{\pi}{j}\right) = \sum_{n_j}^{n_{j+1}} \frac{1}{n\log n}\sin^2\frac{n\pi}{j} + \sum_{k\neq j}\sum_{n_k}^{n_{k+1}} \frac{1}{n\log n}\sin\frac{n\pi}{j}\sin\frac{n\pi}{k}$$

$$= \psi_1+\psi_2,$$

say. Suppose j even. Then, if j is large, there are, between n_j and n_{j+1}, more than $n_{j+1}/2j$ numbers n of the form $(m+\frac{1}{2})j$, where m is an integer ; and the sum

$$\Sigma \frac{1}{n\log n},$$

extended to these values of n, is greater than

$$\frac{1}{4j}\log\left(\frac{\log n_{j+1}}{\log n_j}\right) = \frac{e-1}{4j}\,e^j.$$

Thus

$$(3\,.\,43)\qquad |\psi_1| > \frac{e-1}{4j}\,e^j.$$

On the other hand, the inner sum in ψ_2 is less than a constant multiple of

$$\frac{jk}{|j-k|}\,\frac{1}{n_k\log n_k} \leqslant \frac{jk}{n_k\log n_k},$$

and so

$$(3\,.\,44)\qquad |\psi_2| = O\left(j\Sigma\frac{k}{n_k\log n_k}\right) = O(j).$$

Finally, from (3 . 42)–(3 . 44), it follows that

$$\psi\left(\frac{\pi}{j}\right) = \psi_1+\psi_2 \to \infty,$$

which proves the theorem.

It is hardly necessary to point out that Theorem V greatly increases

the interest of Theorem **T**, and indeed of Theorem 50 of our paper **1**, of which Theorem **T** is a generalisation.

3.5. We can easily adapt the example by which we proved Theorem **V**, so as to prove

THEOREM **W**.—*It is possible to find a convergent series* Σa_n *and a regular path* C, *so that* $\Phi(x)$ *does not tend to a limit when* x *tends to* 1 *along* C.

We take $$a_n = (n+1)\, b_n,$$

where b_n is the a_n of the last paragraph. The argument by which we proved Σb_n convergent is sufficient to establish the convergence of Σa_n.*

Also $$\Phi(x) = \frac{1}{1-x}\,\Sigma b_n(1-x^{n+1}) = \frac{B-xg(x)}{1-x},$$

where $B = \Sigma b_n$ and $g(x) = \Sigma b_n x^n$. And as $g(x)$ does not tend to a limit, $\Phi(x)$ does not do so.

This proves the theorem. It will be observed that there is a great deal to spare in the conclusion: we have proved, in fact, that $\Phi(x)$ assumes values of order greater than that of $1/|1-x|$. The fact is that it ought to be possible to prove that series exist which satisfy the conditions of Theorem **W** and whose coefficients are of order $\phi(n)/n$, where $\phi(n)$ is any function which tends to infinity with n. Such an example would (in conjunction with Theorem **V**) prove that Theorem **T** is a best possible theorem in the same sense as (*e.g.*) Theorem **F**, and that neither hypotheses nor conclusion can be improved upon. We have no doubt that this is true, but we have not succeeded in finding an example to prove our point. In our last example the order of a_n is $1/\log n$, which is far from the limit desired. It is therefore not surprising that we should find that our argument carries us some distance over the mark.

At any rate, however, Theorem **W** is enough to show that, in Theorem **T**, some condition beyond that of mere convergence is essential.

* The function $$\phi(n) = \frac{n+1}{n\log n}$$

has the properties (i) that it decreases steadily to the limit zero, and (ii) that

$$\Sigma j\phi(n_j)$$

is convergent. These were the only properties of $1/(n\log n)$ used in the proof of the convergence of Σb_n.

4.

FOURIER SERIES.

Proof of Theorem **X.**

4 . 1. It has been proved by Fatou* that, *if*

$$a_n = o\left(\frac{1}{n}\right), \qquad b_n = o\left(\frac{1}{n}\right), \tag{4 . 11}$$

then the necessary and sufficient condition that the series

$$\tfrac{1}{2}A_0 + \Sigma A_n = \tfrac{1}{2}a_0 + \Sigma (a_n \cos n\theta + b_n \sin n\theta), \tag{4 . 12}$$

which is certainly the Fourier series of a summable function $f(\theta)$†, *should converge to the sum* A, *is that*

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A \tag{4 . 13}$$

when $\alpha \to 0$, *or* (what is the same thing) *that*

$$\Sigma A_n \frac{\sin n\alpha}{n\alpha} \to A \tag{4 . 14}$$

when $\alpha \to 0$.

---

* Fatou, *l.c.* Fatou does not state the whole of this result explicitly as one theorem, but it is contained in pp. 345–7, 385–7 of his memoir. It is important to observe that, if the conditions (4 . 11) are satisfied and $F(\theta)$ is the integral of $f(\theta)$—or, what is the same thing, the sum of the series obtained by integrating (4 . 12) term-by-term—then

$$F(\theta+\alpha) + F(\theta-\alpha) - 2F(\theta) = o(\alpha)$$

for every value of θ, in virtue of a well known theorem of Riemann (quoted by Fatou, *l.c.*, p. 385). It follows that the three formulæ

$$\frac{1}{\alpha}\int_{\theta}^{\theta+\alpha} f(t)\,dt \to A, \qquad \frac{1}{\alpha}\int_{\theta-\alpha}^{\theta} f(t)\,dt \to A, \qquad \frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A$$

are equivalent. This ceases to be true when the conditions (4 . 11) are replaced by the more general conditions (4 . 21), as appears at once from the simple example of the series

$$\Sigma \frac{\sin n\theta}{n}.$$

† By the "Riesz-Fischer Theorem", $\Sigma(a_n^2 + b_n^2)$ being convergent. In fact

$$\Sigma(|a_n|^{1+\delta} + |b_n|^{1+\delta})$$

is convergent for every positive δ, from which it follows that $|f(\theta)|^k$ is summable for all positive values of k. See W. H. Young, "On the determination of the summability of a function by means of its Fourier constants", *Proc. London Math. Soc.*, Ser. 2, Vol. 12, 1912, pp. 71–88.

The investigations of Sections 2 and 3 suggest very forcibly that it should be possible to replace the conditions (4 . 11) of Fatou's theorem by the corresponding conditions of the "O" type. We proceed to prove that this is so.

4 . 2. THEOREM **X**.*—*If a_n and b_n are the Fourier constants of a summable function $f(\theta)$, and*

$$a_n = O\left(\frac{1}{n}\right), \qquad b_n = O\left(\frac{1}{n}\right), \tag{4 . 21}$$

then the necessary and sufficient condition that the Fourier series of $f(\theta)$ should converge to the sum A is that

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A \tag{4 . 22}$$

when $\alpha \to 0$.

We may plainly take $A = 0$ and suppose that $|nA_n| < 1$.

In the first place, the condition is sufficient. This may be proved in a variety of manners by a mere combination of known theorems.

(*a*) We may prove, by using Poisson's integral in precisely the same way as Fatou, that

$$\Sigma A_n r^n \to 0,$$

when $r \to 1$. The convergence of ΣA_n then follows from Theorem **F**.†

(*b*) It was proved by Lebesgue‡ that the Fourier series of $f(\theta)$ is summable (C, 2), to sum A, for any value of θ for which (4 . 22) is satisfied. The result then follows from the theorem that a series whose general term is of order $1/n$ cannot be summable by Cesàro's means unless it is convergent.§

---

* We have already published a proof of this theorem, by a different method, in the *Comptes rendus* of December 24th, 1917 (our paper **5**).

† Fatou, of course, uses Theorem **D**.

‡ H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, Vol. 61, 1905, pp. 251–280 (p. 278).

§ G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320.

(*c*) It has been shown by W. H. Young* that if the Fourier series of

$$g(a) = \frac{1}{\sin \frac{1}{2}a} \int_{\theta-a}^{\theta+a} f(t)\,dt$$

is summable (C, r) for $a = 0$, then the Fourier series of $f(\theta)$ is summable $(C, r+1)$. If (4 . 23) is satisfied, $g(a)$ is continuous for $a = 0$, and we can take $r = 1$. The proof may then be completed as under (*b*).

4 . 3. We have now to prove that the condition is also necessary. We write†

$$\Phi = \sum_1^\infty A_n \frac{\sin na}{na} = \sum_1^m + \sum_{m+1}^\infty = \Phi_1 + \Phi_2, \tag{4 . 31}$$

say, where

$$m = [K/a],$$

so that $K/2a < m \leqslant K/a$ for all sufficiently small values of a. Then

$$|\Phi_2| < \frac{1}{a} \sum_{m+1}^\infty \frac{1}{n^2} < \frac{1}{ma} < \frac{2}{K}. \tag{4 . 32}$$

Again, if we write

$$t_n = \sum_n^\infty \frac{A_\mu}{\mu},$$

we have (as in § 3 . 2)

$$|t_n| < \frac{\mathbf{m}}{n}, \quad |t_n| < \frac{\delta}{n}\ (n \geqslant n_0), \quad \left|\sum_1^n t_\mu\right| < \delta\ (n \geqslant n_1). \tag{4 . 33}$$

Now

$$\begin{aligned} \Phi_1 = \sum_1^m A_n \frac{\sin na}{na} &= \frac{1}{a} \sum_1^m (t_n - t_{n+1}) \sin na \\ &= \frac{\sin a}{a} \sum_1^m t_n - \frac{\sin a}{a} \sum_1^m t_n \{1 - \cos(n-1)a\} \\ &\qquad - \frac{1-\cos a}{a} \sum_1^m t_n \sin(n-1)a - \frac{1}{a} t_{m+1} \sin ma \\ &= \Phi_1' + \Phi_1'' + \Phi_1''' + \Phi_1'''', \end{aligned} \tag{4 . 34}$$

---

* W. H. Young, "On the Convergence of a Fourier series and its allied series", *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 254–272 (pp. 262–266). Young's argument depends only on a series of elementary identities, and includes a new and greatly simplified proof of Lebesgue's theorem quoted above.

† The proof which we give here is (if hardly shorter) considerably simpler in principle than that which we gave in **5**. Naturally Theorem **T** can also be proved by an adaptation of our former method.

say. In the first place

$$|\Phi_1''''| < \frac{\delta}{ma} < \frac{2\delta}{K}, \tag{4.35}$$

if $m \geqslant n_0$, and

$$|\Phi_1'| < \delta, \tag{4.36}$$

if $m > n_1$; and each of these conditions is satisfied when a is sufficiently small. Secondly,

$$|\Phi_1'''| < \tfrac{1}{2}a \sum_1^m |t_n| < a\mathbf{m} \log m < a\mathbf{m} \log \frac{K}{a}. \tag{4.37}$$

Finally,

$$|\Phi_1''| < \tfrac{1}{2}a^2 \sum_1^m n^2 |t_n| = \tfrac{1}{2}a^2 \left(\sum_1^{n_0-1} + \sum_{n_0}^m \right) n^2 |t_n| \tag{4.38}$$

$$< \tfrac{1}{2}a^2\mathbf{m}n_0^2 + \tfrac{1}{2}a^2\delta m^2 < \tfrac{1}{2}a^2\mathbf{m}n_0^2 + \tfrac{1}{2}K^2\delta.$$

From (4.31), (4.32), and (4.34)–(4.38), we deduce

$$|\Phi| < \frac{2}{K} + \tfrac{1}{2}K^2\delta + \delta + \frac{2\delta}{K} + a\mathbf{m} \log \frac{K}{a} + \tfrac{1}{2}a^2\mathbf{m}n_0^2. \tag{4.39}$$

Given ϵ, we choose $K(\epsilon)$ so that $2/K < \frac{1}{3}\epsilon$; then $\delta(\epsilon, K) = \delta(\epsilon)$ so that

$$\tfrac{1}{2}K^2\delta + \delta + \frac{2\delta}{K} < \tfrac{1}{3}\epsilon;$$

and then $\eta = \eta(\epsilon, K, \delta) = \eta(\epsilon)$ so that (4.39) is satisfied, and

$$a\mathbf{m} \log \frac{K}{a} + \tfrac{1}{2}a^2\mathbf{m}n_0^2 < \tfrac{1}{3}\epsilon,$$

if $0 < a < \eta$. We have then $|\Phi| < \epsilon$,

if $0 < a < \eta$, and the theorem is proved.

4.4. The condition (4.22) is certainly satisfied if $F(\theta)$ has a differential coefficient equal to A. But the convergence of ΣA_n does *not* involve the existence of such a differential coefficient. Thus, if

$$f(\theta) = \Sigma \frac{\sin n\theta}{n},$$

and we consider the particular value $\theta = 0$, the sum of the series is zero, and

$$\frac{1}{2a} \int_{-a}^{a} f(t)\, dt \to 0.$$

But, if a is positive,

$$\frac{1}{a}\int_0^a f(t)\,dt \to \tfrac{1}{2}\pi, \qquad \frac{1}{a}\int_{-a}^0 f(t)\,dt \to -\tfrac{1}{2}\pi.$$

In this respect Theorem **X** differs essentially from Fatou's theorem of which it is the generalisation.

Supplementary Remarks.

4.5. We shall conclude the paper with a few theorems of a slightly different character.

The characteristic properties of series which satisfy the condition (o) or (O) are shared to a great extent by series Σa_n such that

$$\Sigma n\,|a_n|^2, \tag{4.51}$$

or, more generally,

$$\Sigma n^p\,|a_n|^{p+1} \quad (p \geqslant 1), \tag{4.52}$$

are convergent.* It is easy to see that this is true of the properties which have been discussed in this paper.

Let us suppose, for example, that the series (4.51) is convergent; and let us return to the proof of Lemma a.

We can choose m so that

$$\sum_m^\infty n\,|a_n|^2 < \epsilon. \tag{4.53}$$

This being so, we have

$$|\phi_1| \leqslant \frac{2}{|1-x|}\sum_\nu^\infty \frac{|a_n|}{n+1} \leqslant \frac{2}{|1-x|}\sqrt{\left(\sum_\nu^\infty n\,|a_n|^2 \sum_\nu^\infty \frac{1}{n(n+1)^2}\right)}$$

$$\leqslant \frac{2\sqrt{\epsilon}}{\nu\,|1-x|} < 4\sqrt{\epsilon}.$$

* See L. Fejér, "Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf der schlichte Ebene", *H. A. Schwarz Festschrift*, 1914, pp. 42–53; and our paper **3**. It is to be observed that the theorems which depend upon a condition of this type have all the simpler "o" character, and their proofs do not involve the peculiar difficulties of those of the "O" theorems.

Also

$$(4.54) \qquad |s_{\nu-1}-\phi_1| \leqslant \tfrac{1}{2}|1-x| \sum_{1}^{m} n|a_n| + \tfrac{1}{2}|1-x| \sum_{m+1}^{\nu-1} n|a_n|.$$

The first term on the right hand side of (4.54) is less than $\sqrt{\epsilon}$ if x is near enough to 1. The second does not exceed

$$\tfrac{1}{2}|1-x| \sqrt{\left(\sum_{m}^{\nu-1} n|a_n|^2 \sum_{m}^{\nu-1} n\right)} \leqslant \tfrac{1}{2}\nu|1-x| \sqrt{\epsilon} < \sqrt{\epsilon}.$$

Hence $$|s_{\nu-1}-\Phi(x)| < 6\sqrt{\epsilon},$$

if x is near enough to 1. As C is continuous, ν passes through an unbroken sequence of integral values as $x \to 1$. We thus obtain

THEOREM **U**1. — *If* $\Sigma n|a_n|^2$ *is convergent, then the necessary and sufficient condition that* Σa_n *should be convergent is that* $\Phi(x)$ *should tend to a limit when* x *tends to* 1, *either along any particular path* C, *or along all.*

There is no difficulty in proving the more general result which holds when the series (4.52) is convergent. It is only necessary to use the generalised form of the Cauchy-Schwarz inequality.*

Similarly we have

THEOREM **X**1. — *If* $\Sigma n a_n^2 + b_n^2$ *is convergent, then the necessary and sufficient condition for the convergence of the series*

$$\tfrac{1}{2}a_0 + \Sigma (a_n \cos n\theta + b_n \sin n\theta) \sim f(\theta)$$

is that

$$\frac{1}{2\alpha} \int_{\theta-\alpha}^{\theta+\alpha} f(t)\, dt$$

should tend to a limit when $\alpha \to 0$.

4.6. We have found that the condition (4.13) is both necessary and sufficient for the convergence of two important classes of Fourier series. There is a third class for whose convergence it is a *necessary*, though not a sufficient condition. This is the class of *Fourier series of bounded functions*.

---

* Cf. **3**, p. 136.

We have, in fact,

THEOREM Y.—*If $f(\theta)$ is a summable function, bounded in the neighbourhood of a particular value of θ, and if the Fourier series of $f(\theta)$ is convergent for that value of θ, then*

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt$$

tends to a limit when $\alpha \to 0$.

Suppose for simplicity that $a_0 = 0$, and let

$$F_1(\theta) = -\Sigma \frac{a_n \cos n\theta + b_n \sin n\theta}{n^2}$$

be the second integral of $f(\theta)$. Then, if the series converges to the sum zero,

$$\phi(\alpha) = F_1(\theta+\alpha)+F_1(\theta-\alpha)-2F_1(\theta) = o(\alpha^2) \tag{4.61}$$

when $\alpha \to 0$, in virtue of another well known theorem of Riemann.*

Now $$\phi'(\alpha) = F(\theta+\alpha)-F(\theta-\alpha) = \int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt.$$

If $f(\theta)$ were everywhere the differential coefficient of its integral, we should have

$$\phi''(\alpha) = f(\theta+\alpha)+f(\theta-\alpha) = O(1),$$

since $f(t)$ is bounded in the neighbourhood of θ; and from (4.61) and the last equation it would follow at once that

$$\phi'(\alpha) = o(\alpha),$$

proving our point. This is not now a valid proof. But it is easy to see that, in such a theorem as

"$\phi(\alpha) = o(\alpha^2)$ *and* $\phi''(\alpha) = O(1)$ *imply* $\phi'(\alpha) = o(\alpha)$",

the second condition may be replaced by the more general condition expressed by the inequality

$$|\phi'(\beta)-\phi'(\gamma)| < K|\beta-\gamma|\ \dagger:$$

---

* See, for example, de la Vallée Poussin, *Cours d'Analyse*, ed. 2, Vol. 2, p. 172.

† See Landau, "Einige Ungleichungen für zweimal Differentiierbare Functionen", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1914, pp. 43–49.

a condition obviously fulfilled in this case, since ϕ' is the integral of a bounded function.

Thus Theorem **Y** is proved. It is, in fact, but a special case of

THEOREM **Z**.—*The Fourier series of a function $f(\theta)$, bounded in the neighbourhood of the particular value of θ under consideration, is either summable by Cesàro means of arbitrarily small positive order, or summable by no Cesàro mean of any order. The necessary and sufficient condition that it should be summable is expressed by the condition* (4 . 13).

The proof of this theorem would, however, carry us too far from the proper subject of the paper.*

* We have published a sketch of the proof, under the title "On the Fourier series of a bounded function", in the *Records of Proceedings at Meetings* for December 6th, 1917 (*Proc. London Math. Soc.*, Vol. 17, p. xiii).

ON THE CONTINUED FRACTIONS CONNECTED WITH THE HYPERGEOMETRIC EQUATION

By E. LINDSAY INCE.

[Received April 24th, 1918.—Read May 9th, 1918.]

1. *Introductory Remarks.*

The infinite continued fractions dealt with in the present paper differ radically from those which most frequently occur in analysis. The latter are usually of the form

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \dots \text{ ad inf.},$$

or of the form

$$\frac{b_1 z}{c_1 z + d_1} + \frac{b_2 z^2}{c_2 z + d_2} + \frac{b_3 z^2}{c_3 z + d_3} + \dots \text{ ad inf.},$$

or again of Stieltjes' form

$$\frac{1}{e_1 z} + \frac{1}{e_2} + \frac{1}{e_3 z} + \frac{1}{e_4} + \dots \text{ ad inf.}$$

These three forms of continued fraction may readily be transformed into one another ; it is therefore sufficient to consider only the first, which is the form into which a power series is most easily converted.* It has been studied, from the point of view of function theory, by Pringsheim and by Van Vleck,† who found that if the coefficients a_1, a_2, ... (which may be complex numbers) tend to a limit k (real or complex), the continued fraction will converge and represent an analytic function over the whole of the z-plane, with the exception of (1) a rectilinear cut drawn from the point $z = -1/4k$ to infinity in the direction away from the

* Heilermann, *Crelle*, Vol. 33 (1845), p. 174 ; Muir, *Proc. London Math. Soc.*, Vol. 7 (1876), p. 112.

† See especially *Trans. Amer. Math. Soc.*, Vol. 5 (1904), p. 253.

origin, and (2) certain isolated points.* The cut in question does not divide the plane into two or more distinct regions, and the same holds for the other two continued fractions into which the first may be transformed.

The case of the continued fractions connected with the hypergeometric equation will, as we have said, be found to differ from the above. The important point of difference is that in their case there exists, for each continued fraction, a *closed* cut which divides the z-plane into distinct regions. Any one continued fraction represents different analytic functions on opposite sides of the associated cut; in other words, the cuts form impassable barriers, beyond which the continued fraction ceases to represent one analytic function, but may represent another.

There are nevertheless other, though analogous, forms of continued fractions associated with the hypergeometric equation which may have different cuts. It results that when one continued fraction ceases to represent a certain function, it may be possible to find another continued fraction which will continue to do so. We have thus a kind of analytical continuation by means of which we are able, to some extent, to carry the representation of a function over the barrier. We shall find, however, that in no case can a function be represented over the entire plane by continued fractions of the type considered.

2. *Origin of the Continued Fractions—their Convergence and Significance.*

The set of differential equations obtained by successive differentiation in the hypergeometric equation

$$z(1-z)\frac{d^2y}{dz^2}+\{\gamma-(\alpha+\beta+1)z\}\frac{dy}{dz}-\alpha\beta y=0 \qquad (1)$$

is equivalent to the set of recurrence relations

$$x_r = b_r x_{r+1}+a_{r+1}x_{r+2} \quad (r=0,\ 1,\ 2,\ \ldots), \qquad (2)$$

in which

$$x_r=\frac{1}{r!}\frac{d^ry}{dz^r},$$

$$b_r=\frac{(r+1)\{\gamma+r-(\alpha+\beta+2r+1)z\}}{(\alpha+r)(\beta+r)},$$

$$a_{r+1}=\frac{(r+1)(r+2)\,z(1-z)}{(\alpha+r)(\beta+r)}.$$

* The non-convergence at isolated points arises merely through the vanishing of the denominator

$$1+\frac{a_2z}{1}+\frac{a_3z}{1}+\ldots.$$

As r increases indefinitely, b_r and a_{r+1} tend to limits which we shall denote by b and a respectively ; we find, in fact, that

$$b = 1-2z,$$

$$a = z(1-z),$$

limits which are independent of the constants α, β, γ of the hypergeometric equation.

We now form the quadratic equation

$$\rho^2 = b\rho + a;$$

assuming for the moment that the roots are of unequal modulus, let the roots be ρ_1, ρ_2, so chosen that $|\rho_1| > |\rho_2|$.

We are now in a position to apply a theorem given by Perron,* viz., *that provided*

$$\overline{\lim}\sqrt[r]{|x_r|} < \frac{1}{|\rho_2|},$$

and further provided that every a_r and at least one x_r be distinct from zero, the following relation holds :

$$\frac{x_1}{x_0} = \frac{1}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots,$$

this infinite continued fraction being uniformly convergent except in the immediate neighbourhood of the zeros of x_0.

In the case we are considering, the roots of the quadratic equation in ρ are $-z$ and $1-z$, so that when $R(z) < \frac{1}{2}$ the root ρ_2 is $1-z$, and when $R(z) > \frac{1}{2}$ the root ρ_2 is $-z$.

A solution x_r of the recurrence-relation (2) is

$$\frac{1}{r!}\frac{d^r}{dz^r}F(\alpha, \beta\,;\ \gamma\,;\ z),$$

where $F(\alpha, \beta\,;\ \gamma\,;\ z)$ is the hypergeometric function.† Another solution of (2) is

$$\frac{1}{r!}\frac{d^r}{dz^r}F(\alpha, \beta\,;\ \alpha+\beta-\gamma+1\,;\ 1-z).$$

These two solutions are analytic functions at all points of the appropriately cut z-plane.

---

* *Die Lehre von den Kettenbrüchen*, p. 291.

† As defined, for example, in Whittaker and Watson's *Modern Analysis*, § 14·5.

In the neighbourhood of a point ξ, the function $F(\alpha, \beta; \gamma; z)$ may be represented by the Taylor's series

$$1+\sum_{r=1}^{\infty} F^{(r)}(\alpha, \beta; \gamma; \xi)\frac{(z-\xi)^r}{r!},$$

whose radius of convergence is $|1-\xi|$. From the Cauchy-Hadamard theorem we deduce that

$$\overline{\lim}\sqrt[r]{\left\{\frac{1}{r!}|F^{(r)}(\alpha, \beta; \gamma; z)|\right\}} = \frac{1}{|1-z|},$$

whence the criterion of convergence is $|1-z| > |\rho_2|$. But we have seen that $|\rho_2| = |z|$ when $R(z) < \frac{1}{2}$; the criterion is then satisfied. Consequently, when $R(z) < \frac{1}{2}$, the continued fraction

$$\frac{1}{b_0}+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots \tag{3}$$

converges, and its value is

$$\frac{F'(\alpha, \beta; \gamma; z)}{F(\alpha, \beta; \gamma; z)} = \frac{d}{dz}\log F(\alpha, \beta; \gamma; z).$$

Applying like reasoning to the function $F(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z)$, we deduce that

$$\overline{\lim}\sqrt[r]{\left\{\frac{1}{r!}|F^{(r)}(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z)|\right\}} = \frac{1}{|z|},$$

whence the criterion of convergence is $|z| > |\rho_2|$. In the region $R(z) > \frac{1}{2}$, in which $|\rho_2| = |1-z|$, this criterion is satisfied. The continued fraction then converges and its value is

$$\frac{d}{dz}\log F(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z).$$

By a simple reduction the continued fraction (3) may be brought into the form

$$\frac{\alpha\beta}{\gamma-(\alpha+\beta+1)z}+\frac{(\alpha+1)(\beta+1)z(1-z)}{\gamma+1-(\alpha+\beta+3)z}+\ldots+\frac{(\alpha+r)(\beta+r)z(1-z)}{\gamma+r-(\alpha+\beta+2r+1)z}+\ldots,$$

which we shall denote by S_1.

Thus we conclude that the continued fraction S_1 converges all over the z-plane, with the exception, first of the rectilinear cut $R(z) = \frac{1}{2}$, and secondly of a certain number of isolated points which are the zeros of $F(\alpha, \beta; \gamma; z)$ if $R(z) < \frac{1}{2}$, or of $F(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z)$ if $R(z) > \frac{1}{2}$.

The foregoing procedure is essentially due to Perron, who, however, only showed that, when $R(z) < \frac{1}{2}$, the continued fraction S_1 represents the logarithmic derivate of $F(\alpha, \beta; \gamma; z)$. That this continued fraction represents, when $R(z) > \frac{1}{2}$, quite another function does not appear to have been so far noted. This is indeed the most striking property of this continued fraction and of the others to be considered in § 3, and distinguishes them in a marked manner from the continued fraction of Heilermann's form. Some of the consequences of this essential peculiarity of the continued fractions associated with the hypergeometric equation will be discussed in the following sections (§§ 3, 4).

3. *The Set of Twelve Continued Fractions connected with the Hypergeometric Equation: the Functions they represent in different parts of the Plane.*

We shall denote the 24 fundamental hypergeometric functions by the symbols* $y_1, y_2, \ldots, y_{24}$. Further, we shall denote by $f_1, f_2, \ldots, f_{24}$, the corresponding logarithmic derivates *with respect to* z. Thus we have so far found a continued fraction S_1 which represents f_1 or f_5 according as $R(z) < \frac{1}{2}$ or $R(z) > \frac{1}{2}$. We shall find that we may likewise group all the functions in pairs such that to each pair there corresponds a single continued fraction.

If, for example, we take the function y_2, we must associate† with it the function y_7. For from y_2 we deduce that

$$f_2 = \frac{\alpha+\beta-\gamma}{1-z} + \frac{d}{dz}\log F(\gamma-\alpha, \gamma-\beta; \gamma; z),$$

and by what has gone before we know that, when $R(z) < \frac{1}{2}$,

$$\frac{d}{dz}\log F(\gamma-\alpha, \gamma-\beta; \gamma; z)$$
$$= \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma-(2\gamma-\alpha-\beta+1)z} + \frac{(\gamma-\alpha+1)(\gamma-\beta+1)z(1-z)}{\gamma+1-(2\gamma-\alpha-\beta+3)z} + \ldots,$$

a continued fraction which converges, when $R(z) > \frac{1}{2}$, to

$$\frac{d}{dz}\log F(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-z).$$

---

* This notation is to be regarded as an extension of that adopted by Forsyth, *Treatise on Differential Equations*, Chap. VI, to denote the 24 series-solutions of the hypergeometric equation.

† It is noteworthy that the first two terms of the bracket in $F(\ldots)$, and also the outside factor, are always the same in the two associated functions.

We therefore conclude that the continued fraction

$$S_2 = \frac{\alpha+\beta-\gamma}{1-z} + \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma-(2\gamma-\alpha-\beta+1)z} + \frac{(\gamma-\alpha+1)(\gamma-\beta+1)z(1-z)}{\gamma+1-(2\gamma-\alpha-\beta+3)z} + \ldots$$
$$+ \frac{(\gamma-\alpha+r)(\gamma-\beta+r)z(1-z)}{\gamma+r-(2\gamma-\alpha-\beta+2r+1)z} + \ldots$$

represents f_2 when $R(z) < \frac{1}{2}$, and f_7 when $R(z) > \frac{1}{2}$.

Similarly, we discover that the continued fraction

$$S_3 = \frac{1-\gamma}{z} + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{2-\gamma-(\alpha+\beta-2\gamma+3)z} + \frac{(\alpha-\gamma+2)(\beta-\gamma+2)z(1-z)}{3-\gamma-(\alpha+\beta-2\gamma+5)z} + \ldots$$
$$+ \frac{(\alpha-\gamma+r+1)(\beta-\gamma+r+1)z(1-z)}{r+2-\gamma-(\alpha+\beta-2\gamma+2r+3)z} + \ldots$$

represents f_3 when $R(z) < \frac{1}{2}$, and f_6 when $R(z) > \frac{1}{2}$, and again, that the continued fraction*

$$S_4 = \frac{1-\gamma}{z} + \frac{\alpha+\beta-\gamma}{1-z} + \frac{(1-\alpha)(1-\beta)}{2-\gamma-(3-\alpha-\beta)z} + \frac{(2-\alpha)(2-\beta)z(1-z)}{3-\gamma-(5-\alpha-\beta)} + \ldots$$
$$+ \frac{(r+1-\alpha)(r+1-\beta)z(1-z)}{r+2-\gamma-(2r+3-\alpha-\beta)z} + \ldots$$

represents f_4 when $R(z) < \frac{1}{2}$, and f_8 when $R(z) > \frac{1}{2}$.

We have thus disposed of four of the solutions of the hypergeometric equation which are appropriate to the singularity $z = 0$, and of four of those appropriate to the singularity $z = 1$, and turn to the eight solutions appropriate to the singularity at infinity and the remaining eight solutions to be associated with them.

Taking first of all y_9 and writing ζ for $1/z$, we have

$$f_9 = -\alpha\zeta - \zeta^2 \frac{d}{d\zeta} \log F(\alpha,\ \alpha-\gamma+1\ ;\ \alpha-\beta+1\ ;\ \zeta).$$

Now the continued fraction

$$\frac{\alpha(\alpha-\gamma+1)}{\alpha-\beta+1-(2\alpha-\gamma+2)\zeta} + \frac{(\alpha+1)(\alpha-\gamma+2)\zeta(1-\zeta)}{\alpha-\beta+2-(2\alpha-\gamma+4)\zeta} + \ldots$$

represents $\dfrac{d}{d\zeta} \log F(\alpha,\ \alpha-\gamma+1\ ;\ \alpha-\beta+1\ ;\ \zeta)$

when $R(\zeta) < \frac{1}{2}$,

* This is the continued fraction which arises out of the set of recurrence-relations derived from the hypergeometric equation by repeated integration. See a paper by the present author, *Proc. Edin. Math. Soc.*, Vol. 34 (1916), p. 146.

and $$\frac{d}{d\xi}\log F(a,\ a-\gamma+1;\ a+\beta-\gamma+1;\ 1-\xi)$$

when $$R(\xi)>\tfrac{1}{2}.$$

But $$f_{21}=-a\xi-\xi^2\frac{d}{d\xi}\log F(a,\ a-\gamma+1;\ a+\beta-\gamma+1;\ 1-\xi).$$

Now when $R(\xi)<\frac{1}{2}$, $|z-1|>1$, and when $R(\xi)>\frac{1}{2}$, $|z-1|<1$. Hence, substituting $1/z$ for ξ and simplifying, we arrive in the end at the result that the continued fraction

$$S_5=-\frac{a}{z}-\frac{1}{z}\cdot\frac{a(a-\gamma+1)}{(a-\beta+1)z-(2a-\gamma+2)}+\frac{(a+1)(a-\gamma+2)(z-1)}{(a-\beta+2)z-(2a-\gamma+4)}+\ldots$$
$$+\frac{(a+r)(a-\gamma+r+1)(z-1)}{(a-\beta+r+1)z-(2a-\gamma+2r)}+\ldots$$

represents f_9 when $|z-1|>1$, and f_{21} when $|z-1|<1$.

By similar processes of reasoning we conclude that the continued fraction

$$S_6=-\frac{\beta}{z}-\frac{1}{z}\cdot\frac{\beta(\beta-\gamma+1)}{(\beta-a+1)z-(2\beta-\gamma+2)}+\frac{(\beta+1)(\beta-\gamma+2)(z-1)}{(\beta-a+2)z-(2\beta-\gamma+4)}+$$
$$\ldots+\frac{(\beta+r)(\beta-\gamma+r+1)(z-1)}{(\beta-a+r+1)z-(2\beta-\gamma+2r)}+\ldots$$

represents f_{10} when $|z-1|>1$, and f_{22} when $|z-1|<1$; that the continued fraction

$$S_7=\frac{a-\gamma}{z}+\frac{a+\beta-\gamma}{1-z}-\frac{1}{z}\cdot\frac{(1-a)(\gamma-a)}{(\beta-a+1)z-(\gamma-2a+2)}$$
$$+\frac{(2-a)(\gamma-a+1)(z-1)}{(\beta-a+2)z-(\gamma-2a+4)}+\ldots+\frac{(r+1-a)(\gamma-a+r)(z-1)}{(\beta-a+r+1)z-(\gamma-2a+2r+2)}+\ldots$$

represents f_{11} when $|z-1|>1$, and f_{23} when $|z-1|<1$; and that the continued fraction

$$S_8=\frac{\beta-\gamma}{z}+\frac{a+\beta-\gamma}{1-z}-\frac{1}{z}\cdot\frac{(1-\beta)(\gamma-\beta)}{(a-\beta+1)z-(\gamma-2\beta+2)}$$
$$+\frac{(2-\beta)(\gamma-\beta+1)(z-1)}{(a-\beta+2)z-(\gamma-2\beta+4)}+\ldots+\frac{(r+1-\beta)(\gamma-\beta+r)(z-1)}{(a-\beta+r+1)z-(\gamma-2\beta+2r+2)}+\ldots$$

represents f_{12} when $|z-1|>1$, and f_{24} when $|z-1|<1$.

The continued fractions S_5 to S_8 cease to converge on the circle

$|z-1|=1$; the remaining four continued fractions S_9 to S_{12} exhibit the same phenomena with regard to the circle $|z|=1$. A process similar to that applied in the case of S_5 leads us to the discovery that the continued fraction

$$S_9=\frac{\alpha}{1-z}+\frac{1}{1-z}\cdot\frac{\alpha(\gamma-\beta)}{-\gamma-(\alpha-\beta+1)z}-\frac{(\alpha+1)(\gamma-\beta+1)z}{-\gamma-1-(\alpha-\beta+3)z}-\ldots$$
$$-\frac{(\alpha+r)(\gamma-\beta+r)z}{--\gamma-r-(\alpha-\beta+2r+1)z}-\ldots$$

represents f_{13} when $|z|>1$, and f_{17} when $|z|<1$.

Likewise we conclude that the continued fraction

$$S_{10}=\frac{\beta}{1-z}+\frac{1}{1-z}\cdot\frac{\beta(\gamma-\alpha)}{-\gamma-(\beta-\alpha+1)z}-\frac{(\beta+1)(\gamma-\alpha+1)z}{-\gamma-1-(\beta-\alpha+3)z}-\ldots$$
$$-\frac{(\beta+r)(\gamma-\alpha+r)z}{--\gamma-r-(\beta-\alpha+2r+1)z}-\ldots$$

represents f_{14} when $|z|>1$, and f_{18} when $|z|<1$; that the continued fraction

$$S_{11}=\frac{1-\gamma}{z}+\frac{\alpha-\gamma+1}{1-z}+\frac{1}{1-z}\cdot\frac{(\alpha-\gamma+1)(1-\beta)}{\gamma-2-(\alpha-\beta+1)z}$$
$$-\frac{(\alpha-\gamma+2)(2-\beta)z}{-\gamma-3-(\alpha-\beta+2)z}-\ldots-\frac{(\alpha-\gamma+r+1)(r+1-\beta)z}{-\gamma-r-2-(\alpha-\beta+r+1)z}-\ldots$$

represents f_{15} when $|z|>1$, and f_{19} when $|z|<1$; and, lastly, that the continued fraction

$$S_{12}=\frac{1-\gamma}{z}+\frac{\beta-\gamma+1}{1-z}+\frac{1}{1-z}\cdot\frac{(\beta-\gamma+1)(1-\alpha)}{\gamma-2-(\beta-\alpha+1)z}$$
$$-\frac{(\beta-\gamma+2)(2-\alpha)z}{-\gamma-3-(\beta-\alpha+2)z}-\ldots-\frac{(\beta-\gamma+r+1)(r+1-\alpha)z}{-\gamma-r-2-(\beta-\alpha+r+1)z}-\ldots$$

represents f_{16} when $|z|>1$, and f_{20} when $|z|<1$.

4. *Relations existing between the Twelve Continued Fractions at Different Parts of the z-Plane.*

It is well known that the 24 fundamental hypergeometric functions are not all independent. They can in fact be grouped into six sets of four, such that the members of each set are equal to or constant mul-

tiples of one another. In like manner the corresponding logarithmic derivates with respect to z can be grouped into six sets each comprising four equal members. We have, in fact, the following relations

1. $f_1 = f_2 = f_{17} = f_{18} = H_1$ say,
2. $f_3 = f_4 = f_{19} = f_{20} = H_2$,,
3. $f_5 = f_6 = f_{21} = f_{22} = H_3$,,
4. $f_7 = f_8 = f_{23} = f_{24} = H_4$,,
5. $f_9 = f_{12} = f_{13} = f_{15} = H_5$,,
6. $f_{10} = f_{11} = f_{14} = f_{16} = H_6$,,

The twelve continued fractions we have obtained converge over the whole of the z-plane with the exception of certain isolated points which represent the zeros of the solution whose logarithmic derivate is in question, and with the exception of the three closed cuts $R(z) = \frac{1}{2}$, $|z-1| = 1$ and $|z| = 1$. Each continued fraction ceases to converge on one only of these cuts and represents on one side of that cut a certain one of the functions H, and on the other side a different function H. The three cuts divide the plane into six distinct regions A, B, C, D, E, F (Fig. 1), within each of which every one of the twelve continued

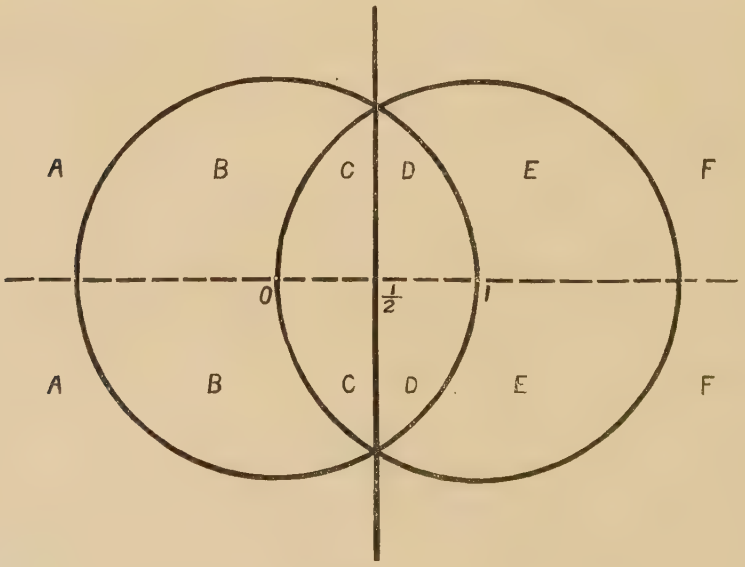


FIG. 1.

fractions represents a definite function, and in each of which the con-

tinued fractions can be grouped into four sets, those in each set being equal to one another within the particular region considered. Thus we have the following sets of relations, viz.,

In region A, $S_1 = S_2 = H_1$, $S_3 = S_4 = H_2$, $S_5 = S_8 = S_9 = S_{11} = H_5$, $S_6 = S_7 = S_{10} = S_{12} = H_6$;

„ B, $S_1 = S_2 = S_9 = S_{10} = H_1$, $S_3 = S_4 = S_{11} = S_{12} = H_2$, $S_5 = S_8 = H_5$, $S_6 = S_7 = H_6$;

„ C, $S_1 = S_2 = S_9 = S_{10} = H_1$, $S_3 = S_4 = S_{11} = S_{12} = H_2$, $S_5 = S_6 = H_3$, $S_7 = S_8 = H_4$;

„ D, $S_1 = S_3 = S_5 = S_6 = H_3$, $S_2 = S_4 = S_7 = S_8 = H_4$, $S_9 = S_{10} = H_1$, $S_{11} = S_{12} = H_2$;

„ E, $S_1 = S_3 = S_5 = S_6 = H_3$, $S_2 = S_4 = S_7 = S_8 = H_4$, $S_9 = S_{11} = H_5$, $S_{10} = S_{12} = H_6$;

„ F, $S_1 = S_3 = H_3$, $S_2 = S_4 = H_4$, $S_5 = S_8 = S_9 = S_{11} = H_5$, $S_6 = S_7 = S_{10} = S_{12} = H_6$.

Here we notice that no equation between any two of the twelve continued fractions holds throughout the whole plane, but each continued fraction is equal, in two of the regions, to three other continued fractions, and in the four other regions to one other continued fraction. Further, no function H is represented over the whole plane by means of the continued fractions, but each H is represented in four regions only.

5. *Logarithmic Cases.*

Exceptional cases arise in three distinct circumstances, viz., $\gamma = 1$, $\gamma = \alpha+\beta$, and $\alpha = \beta$. Similar phenomena arise in all three cases, so that it will be sufficient to consider only the case of $\gamma = 1$. The two solutions

$$y_1 = F(\alpha, \beta;\ \gamma;\ z) \quad \text{and} \quad y_3 = z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1;\ 2-\gamma;\ z),$$

which are distinct if $\gamma \neq 1$, coincide when $\gamma = 1$, and we are led to the "logarithmic case." The continued fraction S_1 now becomes

$$\frac{\alpha\beta}{1-(\alpha+\beta+1)z} + \frac{(\alpha+1)(\beta+1)z(1-z)}{2-(\alpha+\beta+3)z} + \ldots + \frac{(\alpha+r)(\beta+r)z(1-z)}{r+1-(\alpha+\beta+2r+1)z} + \ldots$$

The previous investigation into the convergence of S_1 remains equally valid in the particular case $\gamma = 1$, so that this continued fraction represents, when $R(z) < \frac{1}{2}$, the function

$$F'(\alpha, \beta\,;\ 1\,;\ z)/F(\alpha, \beta\,;\ 1\,;\ z).$$

The continued fraction S_3 now degenerates into S_1, S_4 into S_2, S_{11} into S_9, and S_{12} into S_{10}. The function H_2 also becomes H_1. Thus in this case the continued fraction obtained by repeated integration (S_4) leads to the same solution as that got by repeated differentiation (S_1), so that in the logarithmic case we do not arrive at a complete solution of the equation by the use of continued fractions. This failure to arrive at a complete solution is met with in the case of Legendre's equation.

6. *The Quadratic Transformation.*

An interesting set of relations between continued fractions arises out of the relation

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \alpha & 0 & x \\ \frac{1}{2} & \beta & \frac{1}{2}-\alpha-\beta & \end{Bmatrix} = P\begin{Bmatrix} -1 & \infty & 1 & \\ 0 & 2\alpha & 0 & z \\ \frac{1}{2}-\alpha-\beta & 2\beta & \frac{1}{2}-\alpha-\beta & \end{Bmatrix},$$

where $x = z^2$. In Riemann's notation this is the quadratic transformation due to Kummer.*

The differential equation whose solutions are represented by the P-function on the left-hand side of this equation is

$$x(x-1)\frac{d^2y}{dx^2} + \{-\tfrac{1}{2}+(\alpha+\beta+1)x\}\frac{dy}{dx} + \alpha\beta y = 0, \tag{4}$$

which gives rise directly to the continued fraction

$$\frac{1}{y}\frac{dy}{dx} = \frac{\alpha\beta}{\frac{1}{2}-(\alpha+\beta+1)x} - \frac{(\alpha+1)(\beta+1)x(x-1)}{\frac{3}{2}-(\alpha+\beta+3)x} - \ldots$$
$$- \frac{(\alpha+r)(\beta+r)\,x\,(x-1)}{-\frac{1}{2}(2r+1)-(\alpha+\beta+2r+1)x} - \ldots, \tag{5}$$

or, writing z^2 for x,

$$\frac{1}{2z}\frac{1}{y}\frac{dy}{dz} = \frac{\alpha\beta}{\frac{1}{2}-(\alpha+\beta+1)z^2} - \frac{(\alpha+1)(\beta+1)z^2(z^2-1)}{\frac{3}{2}-(\alpha+\beta+3)z^2} - \ldots$$
$$- \frac{(\alpha+r)(\beta+r)z^2(z^2-1)}{-\frac{1}{2}(2r+1)-(\alpha+\beta+2r+1)z^2} - \ldots \tag{6}$$

* *Crelle*, Vol. 15 (1836), p. 39.

Now, when $R(x) > \frac{1}{2}$, the continued fraction (5) corresponds to that particular solution $F(\alpha, \beta\,;\ \frac{1}{2}-\alpha-\beta\,;\ 1-x)$ of the differential equation (4) which is appropriate to the singularity $x = 1$ and exponent 0. Consequently, when $R(z^2) > 1$, the continued fraction (6) corresponds to the solutions of the transformed equation appropriate to the two singularities $z = 1$ and $z = -1$ with exponent 0.

Let $$z = \xi + i\eta,$$

then $$R(z^2) = \xi^2 - \eta^2 > \tfrac{1}{2}.$$

The region $R(z^2) > \frac{1}{2}$ thus includes those parts of the z-plane to the right of the right-hand branch and to the left of the left-hand branch of the rectangular hyperbola $\xi^2 - \eta^2 = \frac{1}{2}$ (Region I, Fig. 2).

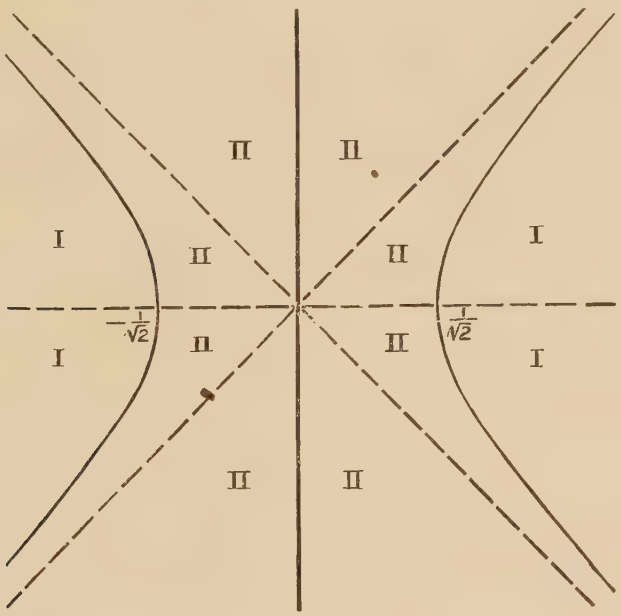


FIG. 2.

The transformed differential equation, which corresponds to the P-function on the right-hand side of the above relation is

$$(z^2-1)\frac{d^2y}{dz^2} + (2\alpha+2\beta+1)\,z\frac{dy}{dz} + 4\alpha\beta y = 0. \qquad (7)$$

This is not a hypergeometric equation, but it is of that type, and can in fact be transformed, by a simple linear substitution in the independent

variable, into a hypergeometric equation. The continued fraction which arises directly out of it is

$$\frac{1}{y}\frac{dy}{dz} = \frac{-4\alpha\beta}{(2\alpha+2\beta+1)z} \; \frac{(2\alpha+1)(2\beta+1)(z^2-1)}{-\;(2\alpha+2\beta+3)\,z} \; -\ldots$$
$$\frac{(2\alpha+r)(2\beta+r)(z^2-1)}{-\;(2\alpha+2\beta+2r+1)\,z} \; -\ldots . \qquad (8)$$

When $R(z) > 0$, this continued fraction converges to the logarithmic derivate of the solution corresponding to the exponent 0 at the singularity $z = +1$, and when $R(z) < 0$ to the logarithmic derivate of the solution corresponding to the exponent 0 at the singularity $z = -1$. It so happens, as is revealed by the symmetry of the P-function, that these two solutions are identical with one another, so that the continued fraction (8) represents a certain odd function over the whole of the z-plane with the exception of the imaginary axis.

These results show us that in Region I (Fig. 2), continued fraction (8) is $2z$ times continued fraction (6). In Region II, (8) retains the significance it had in I, but (6) loses this significance, and there represents the logarithmic derivate of a solution appropriate to the neighbourhood of the origin.

We might, as before, deduce from each of the two differential equations (4) and (7), twelve continued fractions, and then proceed to set up relations between them.

ON HELLINGER'S INTEGRALS

By E. W. Hobson.

[Received June 1st, 1918.—Read June 13th, 1918.]

In connection with the theory of quadratic forms involving an infinite number of variables, which theory is of fundamental importance in relation to integral equations, certain limits occur for the representation of which Hellinger used the notation of integration. These integrals, regarded by Hellinger as a new species of limit, were shewn by Hahn to be reducible to the ordinary integrals defined in the manner introduced into analysis by Lebesgue. In the present communication this reduction is established by a method which will, I think, be found to be appreciably simpler than that employed by Hahn. An extension is also given to a wider class of cases which includes the limits investigated by Hellinger and Hahn as a special case.

1. Let us suppose an interval (a, b) to be divided into a number of parts (a, x_1), (x_1, x_2), ..., (x_{n-1}, b), the greatest of which is $\bar{\delta}^{(1)}$; let a further division of (a, b) be made by dividing some or all of the above parts by the introduction of fresh points of division. Let it be supposed that a system of successive sub-divisions of the interval (a, b) is defined by means of an indefinite continuation of the process. At the m-th stage of the process, let $\bar{\delta}^{(m)}$ be the greatest of the intervals into which (a, b) has been divided. If the system of successive sub-divisions be such that $\bar{\delta}^{(m)}$ converges to zero as $m \sim \infty$, the system will be spoken of as a convergent system of successive sub-divisions of the interval (a, b). Such convergent systems are employed in the definition of the Riemann integral of a limited function. It is convenient to denote the intervals into which (a, b) is divided at the m-th stage by $\delta_1^{(m)}$, $\delta_2^{(m)}$, ..., $\delta_{n_m}^{(m)}$; where n_m is the number of such intervals, and $n_m \sim \infty$ as $m \sim \infty$. In what follows it will frequently be convenient to leave out m, and to denote the intervals by $\delta_1, \delta_2, \ldots, \delta_n$, and the greatest of them by $\bar{\delta}$.

The following theorem will be first established:—

If $f(x)$ denote any function, limited or unlimited, that is summable in (a, b), then, if the square of $f(x)$ is also summable, the expression

$$\frac{1}{\delta_1}\left\{\int_{(\delta_1)} f\,dx\right\}^2+\frac{1}{\delta_2}\left\{\int_{(\delta_2)} f\,dx\right\}^2+\ldots+\frac{1}{\delta_n}\left\{\int_{(\delta_n)} f\,dx\right\}^2$$

converges to $\int_a^b f^2\,dx$, as the intervals $(\delta_1, \delta_2, \ldots, \delta_n)$ into which (a, b) is divided become successively those of any convergent system of sub-divisions of (a, b). If the square of $f(x)$ is not summable, the above sum is divergent.

The integral of $f(x)$ through an interval δ_r is here denoted by $\int_{(\delta_r)} f\,dx$. It may be remarked that, in case $f(x)$ is integrable in accordance with the definition of Riemann, the truth of the theorem is obvious, for $\int_{(\delta_r)} f\,dx$ is in that case equal to δ_r multiplied into a number lying between the upper and lower limits of f in the interval δ_r, and thus the sum is that employed in the definition of $\int_a^b f^2\,dx$.

(*a*) It will first be shewn that the sum

$$\sum_{r=1}^{r=n}\frac{1}{\delta_r}\left\{\int_{(\delta_r)} f\,dx\right\}^2$$

is never diminished as the sub-division of (a, b) passes from any stage to the next. To see this it will be sufficient to shew that, when δ_r is divided into two parts by the introduction of a point of division in the interior of the interval, the two terms which replace $\frac{1}{\delta_r}\left\{\int_{(\delta_r)} f\,dx\right\}^2$ are together greater than or equal to that term. In fact it is sufficient to observe that

$$\frac{(a_1+a_2)^2}{k_1+k_2}\leqslant\frac{a_1^2}{k_1}+\frac{a_2^2}{k_2},$$

which may be written in the form

$$\left(\frac{a_1}{k_1^{\frac{1}{2}}}k_1^{\frac{1}{2}}+\frac{a_2}{k_2^{\frac{1}{2}}}k_2^{\frac{1}{2}}\right)^2\leqslant\left(\frac{a_1^2}{k_1}+\frac{a_2^2}{k_2}\right)(k_1+k_2),$$

and is thus a simple case of a well known inequality.

It has thus been shewn that the numbers

$$\sum_{r=1}^{r=n_m} \frac{1}{\delta_r^{(m)}} \left\{ \int_{(\delta_r^{(m)})} f\,dx \right\}^2,$$

for $m = 1, 2, 3, \ldots$ form a monotone increasing sequence for any convergent system of sub-divisions of (a, b).

Such a monotone sequence of numbers either diverges or else it has a definite limit.

(b) If the theorem holds for each function of a convergent sequence $f_1, f_2, \ldots$ which is monotone increasing, $f_1 \leqslant f_2 \leqslant f_3 \ldots$, and of which all the functions are non-negative for all values of the variable, and which are such that

$$\lim_{n \sim \infty} \int_{(\delta)} f_n(x)\,dx = \int_{(\delta)} f(x)\,dx,$$

and that

$$\lim_{n \sim \infty} \int_a^b \{f_n(x)\}^2\,dx = \int_a^b \{f(x)\}^2\,dx,$$

then the theorem holds for the function f.

We may denote by a_{ms} the number

$$\sum_{r=1}^{r=n_m} \frac{1}{\delta_r^{(m)}} \left\{ \int_{(\delta_r^{(m)})} f_s\,dx \right\}^2.$$

It is then clear that the numbers a_{ms} form a double sequence of numbers such that $a_{m's'} \geqslant a_{ms}$, where $m' \geqslant m$, $s' \geqslant s$. It follows by means of a known theorem that, if either of the repeated limits $\lim_{m \sim \infty} \lim_{s \sim \infty} a_{ms}$, $\lim_{s \sim \infty} \lim_{m \sim \infty} a_{ms}$ exists, the other exists, and their values are the same.

Now, by hypothesis,

$$\lim_{m \sim \infty} a_{ms} = \int_a^b f_s^2\,dx, \qquad \lim_{s \sim \infty} a_{ms} = \sum_{r=1}^{r=n_m} \frac{1}{\delta_r^{(m)}} \left\{ \int_{\delta_r^{(m)}} f\,dx \right\}^2;$$

it then follows that

$$\int_a^b f^2\,dx = \lim_{m \sim \infty} \sum_{r=1}^{r=n_m} \frac{1}{\delta_r^{(m)}} \left\{ \int_{\delta_r^{(m)}} f\,dx \right\}^2;$$

thus the theorem holds for the function f.

(c) The theorem will be proved for the case of a function $f(x)$ defined to have the value 1 at every point of a measurable set e, and the value 0 at all other points of the interval (a, b).

Let $e(x) \equiv \int_a^x f(x)dx$ denote the measure of that part of e which is in the interval (a, x). It is known that $e(x)$ has a differential coefficient $= 1$, at all points of a set $\bar{e}$ contained in e, and of the same measure $m(e)$ as that of e.

If ϵ denote an arbitrarily chosen positive number, $\bar{e}$ contains a closed set E of measure $> m(e)-\epsilon$. If ξ denote any point of E, an interval $(\xi-h, \xi+k)$ can be associated with ξ, such that

$$\frac{\int_\xi^x f\,dx}{x-\xi} > 1-\eta, \quad \text{for} \quad \xi-h \leqslant x \leqslant \xi+k,$$

where η is an arbitrarily chosen positive number (< 1).

In accordance with the Heine-Borel theorem, applied to the closed set of points E, a finite number of these intervals associated with the points of E can be so determined that all the points of E are in the interior of one or more of the intervals of this finite set. The end-points of the intervals of this finite set, together with those points of E with which the intervals are associated, divide (a, b) into a finite number of parts. The sum of those parts which contain points of E in their interiors or at their ends is $> m(e)-\epsilon$. In each of these parts Δ, we have

$$\frac{1}{\Delta}\int_{(\Delta)} f\,dx > 1-\eta.$$

Therefore
$$\Sigma \frac{1}{\Delta}\left\{\int_{(\Delta)} f\,dx\right\}^2 > (1-\eta)^2\,\Sigma\Delta$$
$$> (1-2\eta)\,\{m(e)-\epsilon\}.$$

The positive number ξ being arbitrarily chosen, it is clear that ϵ and η can be so chosen that

$$(1-2\eta)\,\{m(e)-\epsilon\} > m(e)-\xi.$$

It has thus been shewn that (a, b) can be divided into a number of parts such that the corresponding sum is $> m(e)-\xi$, *i.e.* $> \int_a^b f^2\,dx - \xi$.

Also since, in any interval,

$$\frac{1}{\delta}\int_{(\delta)} f\,dx \leqslant 1,$$

we see that $\Sigma \frac{1}{\delta_r}\left\{\int_{(\delta_r)} f\,dx\right\}^2 \leqslant \Sigma \int_\delta f\,dx \leqslant \int_a^b f^2\,dx,$

since $f = f^2$.

A particular system of parts of (a, b) has been shewn to exist, for which the sum is $> \int_a^b f^2 dx - \xi$, and is $\leqslant \int_a^b f^2 dx$. It follows that the upper limit of the sum $\Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} f dx \right\}^2$ for a particular system of sub-divisions of (a, b) into parts is $\int_a^b f^2 dx$.

It will be shewn that the limits of the sum for any two convergent systems of successive sub-divisions of (a, b) are the same, and are consequently equal to $\int_a^b f^2 dx$.

If possible let the limits for two such systems be U, U', where $U' > U$. Consider a certain definite stage of the second system of successive sub-divisions, so far advanced that the corresponding sum is $> U' - \xi$, where ξ is an arbitrarily chosen positive number less than $U' - U$, and let s be the number of intervals into which (a, b) is then divided. Now consider the first system, and let $S(n_m)$ denote the number

$$\sum_{r=1}^{r=n_m} \frac{1}{\delta_r^{(m)}} \left\{ \int_{\delta_r^{(m)}} f dx \right\}^2 ;$$

the greatest interval being $\bar{\delta}^{(m)}$.

If we superimpose on this the sub-division of (a, b) into s parts, in the second system, the increase in $S(n_m)$ caused by the further sub-division so made cannot exceed $s\bar{\delta}^{(m)}$; thus the new sum is less than $S(n_m) + s\bar{\delta}^{(m)}$, or than $U + s\bar{\delta}^{(m)}$.

Now, s being fixed, m may be taken so large, and consequently $\bar{\delta}^{(m)}$ so small, that $U + s\bar{\delta}^{(m)} < U' - \xi$. But this composite sub-division yields a value of the sum $> U' - \xi$, and thus there is a contradiction in assuming that U and U' can be unequal. In virtue of what has been proved above, the value of U or U' can only be $\int_a^b f^2 dx$.

(*d*) Let the function f have the positive values $c_1, c_2, \ldots, c_n$ in the points of the closed sets of points $E_1, E_2, \ldots, E_n$ respectively, and let it have the value 0 at all points of (a, b) not in one or other of these closed sets. We have

$$f = \sum_{r=1}^{r=n} f_r,$$

where f_r is a function which has the value c_r at all points of E_r, and else-

where has the value 0. Since $f_r f_{r'} = 0$, for $r \neq r'$, we see that

$$f^2 = \sum_{r=1}^{r=n} f_r^2.$$

Let us consider the sum

$$\Sigma \frac{1}{\delta} \int_{(\delta)} f_r dx \int_{(\delta)} f_{r'} dx,$$

where $r' \neq r$. Since E_r, $E_{r'}$ are closed sets, there is a minimum distance $d\,(>0)$ between pairs of points one of which belongs to each set. If $\bar{\delta}$, the greatest of the intervals δ, is less than d, then in no interval δ can both $\int_{(\delta)} f_r dx$ and $\int_{(\delta)} f_{r'} dx$ differ from zero; and thus the above sum vanishes. In accordance with (*c*) the theorem to be established holds for each of the functions $f_1, f_2, \ldots, f_n$. If now a sub-division of (a, b) is so far advanced that the greatest interval is less than the minimum distances of all pairs of the closed sets $E_1, E_2, \ldots, E_n$, we find at once, by addition, that the sum

$$\Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} f dx \right\}^2$$

is equal to $\quad \Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} f_1 dx \right\}^2 + \ldots + \Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} f_n dx \right\}^2,$

and this expression converges to

$$\int_a^b f_1^2 dx + \ldots + \int_a^b f_n^2 dx, \quad \text{or to} \quad \int_a^b f^2 dx.$$

The theorem consequently holds for the function f.

(*e*) Next let the function $f(x)$ have the positive values $c_1, c_2, \ldots, c_n$ at the points of the measurable sets $e_1, e_2, \ldots, e_n$ respectively, and let it have the value 0 at any points of (a, b) that do not belong to one or other of these sets.

Let $E_1^{(p)}, E_2^{(p)}, \ldots, E_n^{(p)}$ be closed sets of points contained in $e_1, e_2, \ldots, e_n$ respectively, and of measures greater than $m(e_1) - \eta_p$, $m(e_2) - \eta_p$, $\ldots$, $m(e_n) - \eta_p$, respectively, where η_p is a positive number belonging to a decreasing sequence $\eta_1, \eta_2, \ldots, \eta_p, \ldots$, which converges to zero. The sets $E_r^{(p)}$ can be so defined that $E_r^{(p)}$ is contained in $E_r^{(p+1)}$ for all values of p and of r.

If f_p be the function which has the values $c_1, c_2, \ldots, c_n$ in the sets $E_1^{(p)}, E_2^{(p)}, \ldots, E_n^{(p)}$ respectively, and the value 0 at points which do not belong to any of these sets, we see that $f_1, f_2, \ldots, f_p, \ldots$ is a monotone

increasing sequence such that $\int_{(\delta)} f_p dx$ converges to $\int_{(\delta)} f dx$, and that $\int_a^b f_p^2 dx$ converges to $\int_a^b f^2 dx$. Since the theorem holds for each function f_p, in accordance with (*d*) above, we see, from (*b*) above, that the theorem holds also for the function f.

(*f*) Next, let f be any limited summable function that is everywhere non-negative. In accordance with the definition of a Lebesgue integral a function f_ϵ can be defined such that $0 \leqslant f - f_\epsilon < \epsilon$, where f_ϵ has the values $c_1, c_2, \ldots, c_n$ in a finite set of positive numbers, and is elsewhere zero.

Moreover, if ϵ have successively the values in a diminishing sequence of numbers that converge to zero, the functions $f_{\epsilon_1}, f_{\epsilon_2}, \ldots$ form a monotone increasing sequence which converges to f. Since the totality of all these functions has a finite upper limit, $\int_{(\delta)} f_\epsilon dx$ converges to $\int_{(\delta)} f dx$, and $\int_a^b f_\epsilon^2 dx$ to $\int^b f^2 dx$, as ϵ goes through the sequence. It follows from (*b*) above that the theorem holds for f, since it holds for all the functions f_ϵ in accordance with (*e*).

(*g*) Next let f be an unlimited summable function which is $\geqslant 0$ for all points in (a, b). If $N_1, N_2, \ldots, N_p, \ldots$ be a monotone sequence of increasing positive numbers, such that $N_p \sim \infty$ as $p \sim \infty$, f_{N_p} may be defined to be equal to f at all points x at which $f \leqslant N_p$, and to be equal to N_p when $f > N_p$. In accordance with (*f*) above the theorem holds for the functions f_{N_p} that form a monotone increasing sequence. In accordance with the definition of the Lebesgue integral of an unlimited function, $\int_{(\delta)} f_{N_p} dx$ converges to $\int_{(\delta)} f dx$, and $\int_a^b f_{N_p}^2 dx$ converges to $\int_a^b f^2 dx$, it being assumed that this last integral exists. It now follows by a new application of (*b*) above that the theorem holds for the function f.

In case f is summable but f^2 is not summable, it is clear that, since $\int_a^b f_{N_p}^2 dx$ diverges, so also does the sum corresponding to the function f, for any convergent system of successive sub-divisions of the interval.

(*h*) Lastly, let f be a summable function, whether limited or unlimited, that has both positive and negative values.

The function f may be expressed as the difference $f^{(+)} - f^{(-)}$ of two summable functions $f^{(+)}, f^{(-)}$, each of which is $\geqslant 0$ for all values of x.

Since $\Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} [f^{(+)} + f^{(-)}]\, dx \right\}^2$ converges to $\int_a^b \{f^{(+)} + f^{(-)}\}^2 dx$,

i.e. to $$\int_a^b \{f^{(+)}\}^2 dx + \int_a^b \{f^{(-)}\}^2 dx,$$

and also $\Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} f^{(+)}\, dx \right\}^2$ converges to $\int_a^b \{f^{(+)}\}^2 dx$,

and $\Sigma \frac{1}{\delta} \left\{ \int_{(\delta)} f^{(-)} dx \right\}^2$ converges to $\int_a^b \{f^{(-)}\}^2 dx$,

it follows that $\Sigma \frac{1}{\delta} \int_{(\delta)} f^{(+)}\, dx \int_{(\delta)} f^{(-)}\, dx$ converges to zero.

It now follows that

$$\Sigma \frac{1}{\delta} \left\{ \int_{\delta} (f^{(+)} - f^{(-)})\, dx \right\}^2 \text{ converges to } \int_a^b \{f^{(+)} - f^{(-)}\}^2\, dx\,;$$

and thus the theorem is established generally for every summable function f.

The following theorem which includes the theorem established above may now be stated:—

If $f_1(x)$, $f_2(x)$ be any two functions, limited or unlimited, such that their squares are summable in (a, b), then the expression

$$\frac{1}{\delta_1} \int_{(\delta_1)} f_1 dx \int_{(\delta_1)} f_2 dx + \ldots + \frac{1}{\delta_n} \int_{(\delta_n)} f_1 dx \int_{(\delta_n)} f_2 dx$$

converges to $\int_a^b f_1 f_2 dx$ as the intervals $(\delta_1, \delta_2, \ldots, \delta_n)$ into which (a, b) is divided become successively those of any convergent system of successive sub-divisions of (a, b).

To establish this general theorem, we observe that f_1 and f_2 may be replaced by $f_1^{(+)} - f_1^{(-)}$, $f_2^{(+)} - f_2^{(-)}$ respectively, where these four functions are non-negative.

As the theorem holds for each of the integrals

$$\int_a^b f_1^{(+)} f_2^{(+)} dx, \quad \int_a^b f_1^{(+)} f_2^{(-)} dx, \quad \int_a^b f_1^{(-)} f_2^{(+)} dx, \quad \int_a^b f_1^{(-)} f_2^{(-)} dx,$$

it is clear that it holds also for

$$\int_a^b \{f_1^{(+)} - f_1^{(-)}\}\{f_2^{(+)} - f_2^{(-)}\}\, dx,$$

i.e. for

$$\int_a^b f_1 f_2 dx.$$

Let $g(y)$ be a continuous monotone function of y defined for the interval (α, β) such that as y increases $g(y)$ never diminishes. Let $f(y)$ be a continuous function defined for the interval (α, β), and such that it is constant in any interval contained in (α, β) in which $g(y)$ remains constant.

Let (α, β) be divided into parts (α, y_1), (y_1, y_2), ..., (y_{n-1}, β), where $\alpha = y_0$, $\beta = y_n$, and consider the sum

$$\sum_{r=1}^{r=n} \frac{\{f(y_r) - f(y_{r-1})\}^2}{g(y_r) - g(y_{r-1})},$$

any term being omitted for which the denominator, and consequently also the numerator, vanishes.

In case this sum has a finite upper limit, when all possible modes o dividing (α, β) as above into parts are taken account of, that limit is denoted by Hellinger* by

$$\int_\alpha^\beta \frac{[df(y)]^2}{dg(y)},$$

and may be called a Hellinger's integral.

Let the new variable x be introduced by means of the definition $g(y) = x$; and let $a = g(\alpha)$, $b = g(\beta)$, so that x is in the interval (a, b). The function $f(y)$ may be regarded as a continuous function $F(x)$ of x, the variable y being a uniform function of x. The sum employed in the definition of the Hellinger's integral becomes

$$\sum_{r=1}^{r=n} \frac{\{F(x_r) - F(x_{r-1})\}^2}{x_r - x_{r-1}},$$

where x_r corresponds to y_r.

---

* See his dissertation published at Göttingen in 1907, under the title "Die Orthogonalinvarianten quadratischer Formen von unendlich vielen Veränderlichen." This dissertation has not been accessible to me, but I have followed the indications of its contents given in a paper by Hellinger, published in *Crelle's Journal*, Vol. 136 (1909).

It will be established that :*

The necessary and sufficient condition that $\int_a^\beta \frac{[df(y)]^2}{dg(y)}$ *should exist is that the function* $F(x)[\equiv f(y)]$ *should be the indefinite integral of a function* $\phi(x)$ *of which the square is summable in the interval* (a, b) *of* $x[=g(y)]$. *Moreover*

$$\int_a^\beta \frac{[df(y)]^2}{dg(y)} = \int_a^b \{\phi(x)\}^2 dx,$$

the Hellinger's integral being thus expressed as a Lebesgue integral.

First, to establish the sufficiency of the condition. Let us assume that

$$F(x) = \int_a^x \phi(x)\, dx,$$

where $\int_a^b \{\phi(x)\}^2\, dx$ exists. We have then

$$F(x_r) - F(x_{r-1}) = \int_{x_{r-1}}^{x_r} \phi(x)\, dx,$$

and thence we have, employing the Schwarzian inequality,

$$\{F(x_r) - F(x_{r-1})\}^2 \leqslant (x_r - x_{r-1}) \int_{x_{r-1}}^{x_r} \{\phi(x)\}^2\, dx\ ;$$

and therefore $\sum_{r=1}^{r=n} \frac{\{F(x_r) - F(x_{r-1})\}^2}{x_r - x_{r-1}} \leqslant \int_a^b \{\phi(x)\}^2 dx,$

for any set of sub-divisions of the interval (a, b). It follows that the limit $\int_a^\beta \frac{[df(y)]^2}{dg(y)}$ exists.

Secondly, let the existence of the Hellinger's integral be assumed. It is evident, from the definition, that $\int_a^y \frac{[df(y)]^2}{dg(y)}$, for $a \leqslant y \leqslant \beta$ exists, and that it is a monotone increasing function of y, say $\psi(y)$. We have then, in any interval of y contained in (a, b),

$$\frac{\{\Delta f(y)\}^2}{\Delta g(y)} \leqslant \Delta\psi(y).$$

---

* This theorem was given by H. Hahn, in the *Monatshefte für Math. u. Physik*, Vol. 23 (1912), p. 172.

Thus $$|F(x_r)-F(x_{r-1})| \leqslant [(x_r-x_{r-1})\{\chi(x_r)-\chi(x_{r-1})\}]^{\frac{1}{2}},$$

where $\psi(y) \equiv \chi(x)$. From this we have

$$\sum_{r=1}^{r=n} |F(x_r)-F(x_{r-1})| \leqslant \sum_{r=1}^{r=n} [(x_r-x_{r-1})\{\chi(x_r)-\chi(x_{r-1})\}]^{\frac{1}{2}}$$

$$\leqslant [(b-a)\{\chi(b)-\chi(a)\}]^{\frac{1}{2}};$$

and thus the function $F(x)$ is of limited variation in (a, b); from which it follows that $F'(x)$ exists for almost all values of x in (a, b). In order to shew that $F(x)$ is the indefinite integral of $F'(x)$ it is sufficient* to shew that the variation of $F(x)$ over any set of points of measure zero vanishes. Such a set can be enclosed in a set of non-overlapping intervals (δ) whose measure is arbitrarily small. The variation of $F(x)$ over this set of intervals is, as above, $\leqslant [\Sigma\delta\{\chi(b)-\chi(a)\}]^{\frac{1}{2}}$, where $\Sigma\delta$ can be made as small as we please. The result then follows.

Denoting $F'(x)$ at every point at which it exists, by $\phi(x)$, we may suppose $\phi(x)$ to have the value zero at all points at which $F'(x)$ does not exist. The function $\phi(x)$ being summable in (a, b), the theorem of § 1 is applicable, and it follows that $\int_a^b \phi^2 dx$ exists, and is the limit of the sum

$$\sum_{r=1}^{r=n} \frac{\{F(x_r)-F(x_{r-1})\}^2}{x_r-x_{r-1}},$$

for any convergent system of successive sub-divisions of (a, b). Thus

$$\int_\alpha^\beta \frac{[df(y)]^2}{dg(y)} = \int_a^b \phi^2 dx = \int_a^b \{F'(x)\}^2 dx.$$

If $f_1(y)$, $f_2(y)$ be two continuous functions defined for the interval (α, β), and such that they are constant in any interval in which $g(y)$ remains constant, then in case the sum

$$\sum_{r=1}^{r=n} \frac{\{f_1(y_r)-f_1(y_{r-1})\}\{f_2(y_r)-f_1(y_{r-1})\}}{g(y_r)-g(y_{r-1})}$$

has a definite limit the same for all possible convergent systems of

* See Lebesgue "Sur les intégrales singulières," *Annales de Toulouse*, Ser. 2, Vol. 1 (1909), p. 43.

successive sub-divisions of the interval (α, β) of y, this limit has been denoted by Hellinger by $\int_\alpha^\beta \frac{df_1(y)\, df_2(y)}{dg(y)}$.

Denoting $f_1(y), f_2(y)$ when expressed in terms of $x \equiv g(y)$, by $F_1(x)$, $F_2(x)$, it is sufficient for the existence of the Hellinger's integral $\int_\alpha^\beta \frac{df_1(y)\, df_2(y)}{dg(y)}$ that $F_1(x)$, $F_2(x)$ should be the indefinite integrals of two functions $\phi_1(x)$, $\phi_2(x)$ whose squares are summable in (a, b).

In accordance with the second theorem in § 1, we then see that

$$\int_a^b \phi_1 \phi_2 dx = \int_\alpha^\beta \frac{df_1(y)\, df_2(y)}{dg(y)}.$$

3. In order to extend the theorem of § 1, let us consider the expression

$$\frac{1}{\delta_1^\lambda}\left\{\int_{(\delta_1)} f dx\right\}^{1+\lambda} + \frac{1}{\delta_2^\lambda}\left\{\int_{(\delta_2)} f dx\right\}^{1+\lambda} + \ldots + \frac{1}{\delta_n^\lambda}\left\{\int_{(\delta_n)} f dx\right\}^{1+\lambda},$$

where λ is any positive number. The function f will be assumed to be non-negative for all values of x.

In order to shew that this expression never diminishes as fresh points of sub-division of the interval (a, b) are introduced, it is sufficient to shew that

$$\frac{(a_1+a_2)^{1+\lambda}}{(k_1+k_2)^\lambda} \leqslant \frac{a_1^{1+\lambda}}{k_1^\lambda} + \frac{a_2^{1+\lambda}}{k_2^\lambda}.$$

To prove this we refer to the known inequality

$$\Sigma\alpha\beta \leqslant (\Sigma\alpha^q)^{1/q}\,(\Sigma\beta^p)^{1/p},$$

where p, q are any positive numbers such that $1/p+1/q = 1$. Let

$$\alpha_1 = a_1/k_1^{\lambda/(1+\lambda)}, \quad \alpha_2 = a_2/k_2^{\lambda/(1+\lambda)}, \quad \beta_1 = k_1^{\lambda/(1+\lambda)}, \quad \beta_2 = k_2^{\lambda/(1+\lambda)},$$

$$p = (1+\lambda)/\lambda, \quad q = 1+\lambda;$$

we have then

$$a_1+a_2 \leqslant \left\{\left(\frac{a_1}{k_1^{\lambda/(1+\lambda)}}\right)^{1+\lambda} + \left(\frac{a_2}{k_2^{\lambda/(1+\lambda)}}\right)^{1+\lambda}\right\}^{1/(1+\lambda)} \{k_1+k_2\}^{\lambda/(1+\lambda)},$$

or

$$\frac{(a_1+a_2)^{1+\lambda}}{(k_1+k_2)^\lambda} \leqslant \frac{a_1^{1+\lambda}}{k_1^\lambda} + \frac{a_2^{1+\lambda}}{k_2^\lambda}.$$

We have now the following extension of the theorem of § 1 :—

If $f(x)$ denote any summable function that is non-negative throughout (a, b), and such that $\{f(x)\}^{1+\lambda}$ is summable in (a, b), where λ denotes some positive number, then the expression

$$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\lambda}} \left\{ \int_{(\delta_r)} f\, dx \right\}^{1+\lambda}$$

converges to $\int_a^b f^{1+\lambda}\, dx$, as the intervals $(\delta_1, \delta_2, \ldots, \delta_n)$ into which (a, b) is divided become those of any convergent system of sub-divisions of (a, b). If $f^{1+\lambda}$ is not summable, the above sum is divergent.

When $\lambda = 1$, this is the theorem of § 1.

To prove the theorem it is only necessary to repeat the stages (b), (c), (d), (e), (f), (g) of the proof in § 1; this can be done without any material change in that proof; that (a) is applicable has been proved above.

4. Let it be now assumed that λ is a positive integer. In that case the theorem of § 3 can be freed from the condition that $f \geqslant 0$ in (a, b). Let $1+\lambda = i$, and consider i summable functions $f_1, f_2, \ldots, f_i$, all of which are non-negative in (a, b). It is easily seen that

$$(f_1+f_2+\ldots+f_i)^i - \Sigma(f_2+f_3+\ldots+f_i)^i + \Sigma(f_3+f_4+\ldots+f_i)^i - \ldots \pm \Sigma f^i,$$

is identically equal to a multiple of $f_1 f_2 \ldots f_i$; where the expression in the second term denotes the sum of the i-th powers of the sum of $i-1$ of the f's, that in the third term denotes the sum of the i-th powers of the sum of $i-2$ of the f's, ... For it is clear that the expression vanishes when any one of the f's vanishes. It is thus seen that the product $f_1 f_2 f_3 \ldots f^i$ of the i functions is expressible as the sum of a set of terms, each of which is a positive or negative multiple of the i-th power of the sum of certain of the functions. Let it now be assumed that $f_1^i, f_2^i, \ldots, f_i^i$ are all summable in (a, b). The inequality

$$(f_1+f_2+\ldots+f_i)^i \leqslant i^{i-1}(f_1^i+f_2^i+\ldots+f_i^i),$$

shews that $(f_1+f_2+\ldots+f_i)^i$ is summable in (a, b). It now follows that the product $f_1 f_2 f_3 \ldots f_i$ which has been expressed as the sum of terms each of which is a multiple of the i-th power of the sum of some or all of the functions $f_1, f_2, \ldots, f_i$ is also summable in (a, b). It follows that all the products $f_1^{s_1} f_2^{s_2} \ldots f_i^{s_i}$, when $s_1, s_1, \ldots, s_i$ may have any integral values, including zero, such that their sum is equal to i, are summable; for any such product is that of i functions, some of which are repeated.

Since $$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \left\{ \int_{(\delta_r)} [f_1+f_2+\ldots+f_\iota]\, dx \right\}^\iota$$

converges to $$\int_a^b (f_1+f_2+\ldots+f_\iota)^\iota\, dx,$$

and in this theorem any one or more of the functions $f_1, f_2, \ldots, f_\iota$ may be replaced by zero, and remembering that

$$f_1 f_2 \ldots f_\iota \quad \text{and} \quad \int_{(\delta_r)} f_1 dx \int_{(\delta_r)} f_2 dx \ldots \int_{(\delta_r)} f_\iota dx$$

are expressible as the sums of multiples of ι-th powers of sums of functions chosen from $f_1, f_2, \ldots, f_\iota$, and of the ι-th powers of the integrals of such sums respectively, we see that

$$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \int_{(\delta_r)} f_1 dx \int_{(\delta_r)} f_2 dx \ldots \int_{(\delta_r)} f_\iota dx$$

converges to the value $$\int_a^b f_1 f_2 \ldots f_\iota\, dx.$$

Since any two or more of the functions $f_1, f_2, \ldots, f_\iota$ may be taken to be identical, we see also that

$$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \left\{ \int_{(\delta_r)} f_1 dx \right\}^{s_1} \left\{ \int_{(\delta_r)} f_2 dx \right\}^{s_2} \ldots \left\{ \int_{(\delta_r)} f_\iota dx \right\}^{s_\iota}$$

converges to $$\int_a^b f_1^{s_1} f_2^{s_2} \ldots f_\iota^{s_\iota}\, dx,$$

where $s_1, s_2, \ldots, s_\iota$ are integers (including zero) such that

$$s_1+s_2+\ldots+s_\iota = \iota.$$

Now let f be a function summable in (a, b), but not necessarily everywhere of one sign; we have then $f = f^{(+)} - f^{(-)}$, where $f^{(+)}, f^{(-)}$ are summable functions both of which are non-negative for all the values of x.

Since $$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \left\{ \int_{(\delta_r)} (f^{(+)}+f^{(-)})\, dx \right\}^\iota$$

converges to $$\int_a^b \{f^{(+)}+f^{(-)}\}^\iota\, dx,$$

and since also $$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \left\{ \int_{(\delta_r)} f^{(+)} dx \right\}^s \left\{ \int_{(\delta_r)} f^{(-)} dx \right\}^{\iota-s}$$

converges to $\int_a^b \{f^{(+)}\}^s \{f^{(-)}\}^{\iota-s} dx$,

for $s = 0, 1, 2, 3, \ldots, \iota$, it is seen that

$$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \left\{ \int_{(\delta_r)} (f^{(+)} - f^{(-)})\, dx \right\}^{\iota}$$

converges to $\int_a^b (f^{(+)} - f^{(-)})^{\iota} dx$.

The following theorem has now been established :—

If f be any function summable in (a, b), and if f^{ι} be also summable in (a, b), ι being any positive integer, then

$$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \left\{ \int_{(\delta_r)} f\, dx \right\}^{\iota}$$

converges to $\int_a^b f^{\iota} dx$ as the intervals $(\delta_1, \delta_2, \ldots, \delta_n)$ into which (a, b) is divided become those of any convergent system of sub-divisions of (a, b). If f^{ι} is not summable the above sequence does not converge.

5. If $f_1, f_2, \ldots, f_{\iota}$ be any functions, not necessarily non-negative such that $f_1^{\iota}, f_2^{\iota}, \ldots, f_{\iota}^{\iota}$ are all summable in (a, b), each one of them f_r can be expressed by $f_r^{(+)} - f_r^{(-)}$, the difference of two non-negative functions whose ι-th powers are summable. The product

$$f_1 f_2 \ldots f_{\iota} \quad \text{or} \quad (f_1^{(+)} - f_1^{(-)})(f_2^{(+)} - f_2^{(-)}) \ldots (f_{\iota}^{(+)} - f_{\iota}^{(-)})$$

can be expressed, by multiplying the factors out, as an aggregate of terms to each of which the result established above as regards the product of ι functions all of which are non-negative is applicable.

We see then that the result can be extended to the case of functions which are not necessarily everywhere of one sign. We have the following theorem :—

If $f_1, f_2, \ldots, f_{\iota}$ are functions such that $f_1^{\iota}, f_2^{\iota}, \ldots, f_{\iota}^{\iota}$ are summable in (a, b), then

$$\sum_{r=1}^{r=n} \frac{1}{\delta_r^{\iota-1}} \int_{(\delta_r)} f_1 dx \int_{(\delta_r)} f_2 dx \ldots \int_{(\delta_r)} f_{\iota} dx$$

converges to $\int_a^b f_1 f_2 \dots f_\iota \, dx$, *as the intervals* $(\delta_1, \delta_2, \dots, \delta_r)$ *become those of any convergent system of sub-divisions of* (a, b).

6. Hellinger's definition of the meaning of $\int_a^\beta \frac{[df(y)]^2}{dg(y)}$ given in § 2, may be extended to the case in which

$$\sum_{r=1}^{r=n} \frac{\{f(y_r)-f(y_{r-1})\}^\iota}{\{g(y_r)-g(y_{r-1})\}^{\iota-1}}$$

has a finite upper limit for all possible systems of sub-division of the interval (α, β) into parts $(\alpha, y_1)(y_1, y_2) \dots (y_{n-1}, \beta)$, where $\alpha = y_0$, $\beta = y_n$. The limit, when it exists, may be denoted by $\int_\alpha^\beta \frac{[df(y)]^\iota}{[dg(y)]^{\iota-1}}$, where ι is any positive integer. In this case :—

The necessary and sufficient condition that $\int_\alpha^\beta \frac{[df(y)]^\iota}{[dg(y)]^{\iota-1}}$ *should exist is that the function* $F(x)[\equiv f(y)]$ *should be the indefinite integral of a function* $\phi(x)$ *such that* $\{\phi(x)\}^\iota$ *is summable in* (a, b). *Moreover*

$$\int_\alpha^\beta \frac{[df(y)]^\iota}{[dg(y)]^{\iota-1}} = \int_a^b \{\phi(x)\}^\iota \, dx.$$

To establish the sufficiency of the condition, let it be assumed that

$$F(x) = \int_a^x f(x) \, dx,$$

where $\int_a^b \{\phi(x)\}^\iota \, dx$ exists. We have

$$\{F(x_r)-F(x_{r-1})\}^\iota = \left\{\int_{x_1}^{x_2} \phi(x) \, dx\right\}^\iota \leqslant (x_r-x_{r-1})^{\iota-1} \int_{x_1}^{x_2} \{\phi(x)\}^\iota \, dx,$$

from a known extension of the Schwarzian inequality.

It follows that

$$\sum_{r=1}^{r=n} \frac{\{F(x_r)-F(x_{r-1})\}^\iota}{(x_r-x_{r-1})^{\iota-1}} \leqslant \int_a^b \{\phi(x)\}^\iota \, dx,$$

and thus the limit exists.

Next, let the existence of the limit be assumed; then it is clear that $\int_a^y \frac{[df(y)]^\iota}{[dg(y)]^{\iota-1}}$ exists, and that it is a monotone increasing function $\psi(y)$ of

y; let $\psi(y) = \chi(x)$. We see that, in any interval of y,

$$\frac{\{\Delta f(y)\}^{\iota}}{\{\Delta g(y)\}^{\iota-1}} \leqslant \Delta\psi(y),$$

and thus $\{F(x_r)-F(x_{r-1})\}^{\iota} \leqslant (x_r-x_{r-1})^{\iota-1}\{\chi(x_r)-\chi(x_{r-1})\}$.

It is now shewn, exactly as in § 2, that $F(x)$ is of limited variation in (a, b), and that it is also of limited variation over any set of points of which the measure is zero. Thus $F(x)$ is the indefinite integral of the function $\phi(x)$ which is equal to $F'(x)$ at every point at which $F'(x)$ exists, and it is elsewhere zero.

Lastly, applying the theorem in § 4, we see that $\{\phi(x)\}^{\iota}$ must be summable, in order that the limit may exist, and that the limit is then equal to $\int_a^b \{\phi(x)\}^{\iota}\, dx$.

NOTE ON THE VALUES OF n WHICH MAKE $\frac{d}{dx}\{P_n^{-m}(x)\}$ VANISH AT $x = a$

By H. J. PRIESTLEY.

[Received June 26th, 1918.—Read November 14th, 1918.]

IN a paper "On the Scattering of Sound Waves by a Cone,"* Carslaw uses the fact that the roots of the equation in n, $\frac{d}{dx}P_n^{-m}(x) = 0$ when $x = a$, are real and separate. In a footnote he calls attention to the need for a proof of the theorem.

Such a proof is easily derived from the theory of the Homogeneous Integral Equation.

Consider the differential equation

$$\frac{d}{dt}\left[(1-t^2)\frac{dy}{dt}\right] - m^2(1-t^2)^{-1}y = 0. \tag{1}$$

Solutions are $\{(1-t)/(1+t)\}^{m/2}$, $\{(1+t)/(1-t)\}^{m/2}$.

From these are derived

$$Z_1(t) = \{(1+a)(1-t)/(1-a)(1+t)\}^{m/2} + \{(1-a)(1+t)/(1+a)(1-t)\}^{m/2},$$

$$Z_2(t) = \{(1+a)(1-t)/(1-a)(1+t)\}^{m/2}.$$

Consider now $K(x, t)$ defined by the equations

$$K(x, t) = Z_1(t)\,Z_2(x) \quad (t < x),$$

$$K(x, t) = Z_2(t)\,Z_1(x) \quad (t > x).$$

If $$a < x < 1,$$

$K(x, t)$ is a continuous solution of (1) satisfying the conditions

$$K(x, t) = 0 \quad \text{when} \quad t = 1,$$

$$\frac{\partial}{\partial t}K(x, t) = 0 \quad \text{when} \quad t = a.$$

* *Math. Annalen*, Vol. 75 (1914), p. 143.

Further, $\frac{\partial}{\partial t} K(x, t)$ has a discontinuity at $t = x$, where

$$(1-t^2) \frac{\partial}{\partial t} K(x, t) \Big|_{t=x+}^{t=x-} = 2m.$$

Again, $P_n^{-m}(t)$ is a solution, vanishing at $t = 1$, of

$$\frac{d}{dt}\left[(1-t^2) \frac{d}{dt} P_n^{-m}(t)\right] - m^2(1-t^2)^{-1} P_n^{-m}(t) = -n(n+1) P_n^{-m}(t). \quad (2)$$

Also, if n has one of the required values,

$$\frac{d}{dt} P_n^{-m}(t) = 0 \quad \text{at} \quad t = a.$$

Adopting Hilbert's* well known method we obtain from (1) and (2) the integral equation

$$2mP_n^{-m}(x) = n(n+1) \int_a^1 K(x, t) P_n^{-m}(t)\, dt.$$

If $\lambda_1, \lambda_2, \ldots$ are the characteristic constants of this equation, the required values of n are the roots of the set of quadratic equations

$$n^2+n = 2m\lambda_s \quad (s = 1, 2, 3, \ldots).$$

Now $K(x, t)$ is a symmetric function of x and t, consequently λ_s is real.

It follows that possible values of n are real or conjugate complex quantities.

The argument used by Macdonald† in his discussion of the roots of

$$P_n^{-m}(a) = 0,$$

shows that conjugate complex roots or multiple roots are impossible.

Hence the roots are real and separate.

---

* See Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," Chap. VII, or *Gött. Nachr.*, 1904, pp. 213 *et seq.*

† *Proc. London Math. Soc.*, Ser. 1, Vol. XXXI (1899), pp. 265–266.

A DETAIL IN CONFORMAL REPRESENTATION

By J. HODGKINSON.

[Received October 14th, 1918.—Read November 14th, 1918.]

I. *Introductory.*

The writer has discussed certain aspects of the conformal representation of a curvilinear triangle on a previous occasion. It was then shown with regard to those triangles whose sides have a real common orthogonal circle that, in order that the function z, which effects the representation upon a half plane of a curvilinear triangle in the w plane, may have a finite number of branches, the triangle must be a reduced triangle; and further that it must be built up of symmetrical repetitions of a triangle whose angles are $\pi/2$, π/l, π/m, where l, m are integers. It was also shown how in any particular case the possibility or otherwise of such a composition can be determined by a finite number of tests.*

It will now be shown that the total number of such compositions is ten.

The method employed can be applied equally to the cases of those triangles whose sides are rectilinear or have an imaginary common orthogonal circle. These cases, however, have been fully discussed otherwise.† I have also discussed the case in which one or more of the angles of the triangle is zero.

II. *The Triangle with Finite Angles.*

Suppose we have a fundamental triangle whose angles are $\pi/2$, π/p, π/q. The sum of the angles of any composite triangle built up of symmetrical repetitions of this fundamental triangle is

$$\frac{l\pi}{2}+\frac{m\pi}{p}+\frac{n\pi}{q},$$

where l, m, n are integers. Since we have only to deal with reduced

* *Proc. London Math. Soc.*, Ser. 2, Vol. 15 (1916), pp. 166–181.

† Schwarz, *Journal für Mathematik*, Vol. 75 (1873), pp. 292–305; Burnside, *Proc. London Math. Soc.*, Ser. 1, Vol. XXIV (1893), pp. 187–206.

triangles, the sum of whose angles is less than π, l must be less than 2, m than p, n than q. We note further that $l+m+n$ is not less than 3.

Let the composite triangle contain k repetitions of the original triangle. We suppose that the vertices lie on the sides and in the interior of the composite triangle in the following manner: there are a_1 repetitions of the angle $\pi/2$ on the sides and a_2 in the interior, b_1 repetitions of the angle π/p on the sides and b_2 in the interior, c_1 repetitions of the angle π/q on the sides and c_2 in the interior.

Then
$$l+2(a_1+2a_2) = k,$$
$$m+p(b_1+2b_2) = k,$$
$$n+q(c_1+2c_2) = k;$$

for at each of the a_1 points there are two repetitions of the angle $\pi/2$, at each of the a_2 points four repetitions, and so on.

Next let us consider the closed polyhedron consisting of the triangle and its reverse face, and apply Euler's theorem to the number of its faces, vertices, and edges.

The number of edges is $3k$, the number of faces $2k$. The total number of vertices is
$$3+a_1+2a_2+b_1+2b_2+c_1+2c_2.$$

Euler's theorem then gives
$$a_1+2a_2+b_1+2b_2+c_1+2c_2 = k-1.$$

We write $\quad a_1+2a_2 = a, \quad b_1+2b_2 = b, \quad c_1+2c_2 = c.$

We have now to solve in positive integers
$$2a+l = k, \quad pb+m = k, \quad qc+n = k,$$
$$a+b+c = k-1,$$

with the conditions
$$l < 2, \quad m < p, \quad n < q, \quad l+m+n \geqslant 3.$$

We first of all note that, since
$$\frac{1}{2}+\frac{1}{p}+\frac{1}{q}$$

is less than 1, then, assuming p not greater than q, q is not less than 7 when $p = 3$, and q is not less than 5 when $p = 4$.

From the given equations we deduce
$$(p-4)\,b+(q-4)\,c+(l+m+n) = 4-l.$$

p be not less than 5, the only possible solution is

$$k = 2, \quad a = 1, \quad b = 0, \quad c = 0, \quad m = 2, \quad n = 2.$$

We write $p = 4$, and $k = 4\lambda$, $4\lambda+1$, $4\lambda+2$, $4\lambda+3$ in turn, and find three solutions.

Suppose, *e.g.*, $k = 4\lambda+3$. Then $a = 2\lambda+1$, $b = \lambda$, whence $c = \lambda+1$, $l = 1$, $m = 3$; and

$$q(\lambda+1)+n = 4\lambda+3,$$

or

$$(q-4)(\lambda+1)+n+1 = 0,$$

which has no solutions in positive integers.

Similarly we write $p = 3$, and $k = 6\lambda$, $6\lambda+1$, $6\lambda+2$, $6\lambda+3$, $6\lambda+4$, $6\lambda+5$ in turn, and find thirteen solutions.

The full list of solutions is

k	p	q	a	b	c	l	m	n
2	p	q	1	0	0	0	2	2
3	3	q	1	1	0	1	0	3
4	3	q	2	1	0	0	1	4
4	4	q	2	1	0	0	0	4
6	3	q	3	2	0	0	0	6
6	4	5	3	1	1	0	2	1
8	3	7	4	2	1	0	2	1
8	4	5	4	2	1	0	0	3
9	3	7	4	3	1	1	0	2
10	3	7	5	3	1	0	1	3
10	3	8	5	3	1	0	1	2
12	3	7	6	4	1	0	0	5
12	3	8	6	4	1	0	0	4
12	3	9	6	4	1	0	0	3
16	3	7	8	5	2	0	1	2
18	3	7	9	6	2	0	0	4
24	3	7	12	8	3	0	0	3

These solutions however do not all lead to possible composite triangles, while some lead to two distinct triangles. This is because we have as yet taken no account of the manner in which a is split up into its two parts a_1 and $2a_2$.

The inquiry however is unnecessary, for we have figures which decide the possibility of composition or otherwise upon inspection.*

* For $p = 3$, $q = 7$, see Klein-Fricke, *Theorie der Elliptischen Modulfunctionen*, Vol. 1, p. 109; for $p = 4$, $q = 5$, see Schwarz, *l.c.*

The roughest of figures is sufficient to demonstrate the impossibility of the composition of a triangle $(\pi/9, \pi/9, \pi/9)$, by means of twelve repetitions of a triangle $(\pi/2, \pi/3, \pi/9)$. For, since $c = 1$, nine repetitions of the fundamental triangle have a common vertex on one of the sides of the attempted composite triangle. These form a hexagon and it is manifestly impossible to convert this into a triangle by adding three more repetitions.

In the same way we satisfy ourselves of the impossibility of building up the triangle $(\pi/3, \pi/8, \pi/8)$ by means of ten repetitions of the triangle $(\pi/2, \pi/3, \pi/8)$.

In the other case in which the triangle $(\pi/2, \pi/3, \pi/8)$ occurs, by means of six repetitions we build up the triangle $(\pi/2, \pi/8, \pi/8)$. This being itself repeated we get the triangle $(\pi/4, \pi/8, \pi/8)$.

From the figures, then, and these last considerations we arrive at the following possible compositions:

Fundamental Triangle.	*Composite Triangle.*	k.
$\left(\frac{\pi}{2}, \frac{\pi}{p}, \frac{\pi}{q}\right)$	$\left(\frac{2\pi}{p}, \frac{\pi}{q}, \frac{\pi}{q}\right)$	2
	$\left(\frac{2\pi}{q}, \frac{\pi}{p}, \frac{\pi}{p}\right)$	2
$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{q}\right)$	$\left(\frac{\pi}{2}, \frac{2\pi}{q}, \frac{\pi}{q}\right)$	3
	$\left(\frac{\pi}{3}, \frac{3\pi}{q}, \frac{\pi}{q}\right)$	4
	$\left(\frac{2\pi}{q}, \frac{2\pi}{q}, \frac{2\pi}{q}\right)$	6
	$\left(\frac{4\pi}{q}, \frac{\pi}{q}, \frac{\pi}{q}\right)$	6
$\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{q}\right)$	$\left(\frac{2\pi}{q}, \frac{\pi}{q}, \frac{\pi}{q}\right)$	4
$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}\right)$	$\left(\frac{\pi}{3}, \frac{2\pi}{7}, \frac{\pi}{7}\right)$	10
	$\left(\frac{\pi}{7}, \frac{\pi}{7}, \frac{\pi}{7}\right)$	24
$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{8}\right)$	$\left(\frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{8}\right)$	12

Certain of the results which appear are, in a sense, accidental. Thus

from the triangle $(\pi/2, \pi/4, \pi/q)$, we obtain the triangle $(\pi/2, \pi/q, \pi/q)$; and this triangle also being right-angled is capable of being used for further composition by means of which we get the triangle $(2\pi/q, \pi/q, \pi/q)$. Similarly from the triangle $(\pi/2, \pi/3, \pi/8)$, we obtained the triangle $(\pi/2, \pi/8, \pi/8)$ which again led to the triangle $(\pi/4, \pi/8, \pi/8)$. These appeared in the solution of the indeterminate equations.

In a similar manner we sometimes find that the sets of triangles derived from two fundamental triangles have a triangle in common.

Thus the set derived from the triangle $(\pi/2, \pi/3, \pi/9)$ is

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{9}\right), \quad \left(\frac{2\pi}{3}, \frac{\pi}{9}, \frac{\pi}{9}\right), \quad \left(\frac{2\pi}{9}, \frac{\pi}{3}, \frac{\pi}{3}\right), \quad \left(\frac{\pi}{2}, \frac{2\pi}{9}, \frac{\pi}{9}\right),$$

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{9}\right), \quad \left(\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{2\pi}{9}\right), \quad \left(\frac{4\pi}{9}, \frac{\pi}{9}, \frac{\pi}{9}\right),$$

while the set derived from the triangle $(\pi/2, \pi/3, \pi/18)$ is

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{18}\right), \quad \left(\frac{\pi}{9}, \frac{\pi}{3}, \frac{\pi}{3}\right), \quad \left(\frac{2\pi}{3}, \frac{\pi}{18}, \frac{\pi}{18}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{9}, \frac{\pi}{18}\right),$$

$$\left(\frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{18}\right), \quad \left(\frac{\pi}{9}, \frac{\pi}{9}, \frac{\pi}{9}\right), \quad \left(\frac{2\pi}{9}, \frac{\pi}{9}, \frac{\pi}{9}\right).$$

The triangle $(\pi/3, \pi/3, \pi/9)$ is common to both sets. As a consequence the triangle functions derived from any two of these triangles can be connected by an algebraic relation.

Similarly the sets derived from the triangles $(\pi/2, \pi/3, \pi/7)$ and $(\pi/2, \pi/3, \pi/14)$ each contain the triangle $(\pi/7, \pi/7, \pi/7)$.

III. *The Triangle with some of its Angles Zero.*

If we apply the same methods to the conformal representation of triangles with zero angles our equations become

$$2a+l=k, \quad pb+m=k, \quad n=k,$$

$$a+b=k-1.$$

The number c disappears, for we cannot build up an angle π or 2π by means of a finite number of repetitions of zero. The possible solutions are the same as those of the general case in which q has an arbitrary value; in this instance q is made infinite.

The number of triangles which have what we called an accidental connection is considerable; for, retaining only those which give rise to

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1919.

Liabilities.	£	s.	d.	Assets.	£	s.	d.
Lieut.-Col. Campbell's Fund	499	7	5	London & North Western Railway £385 Four per Cent. Guaranteed Stock at cost ...	499	7	5
Life Compositions Fund	2385	5	0	India Stock £700. 7*s.* 1*d.* 3 per Cent. at cost	700	0	0
Invested Surplus Fund	700	0	0	War Stock £250 5 per Cent. 1929–47 at cost	236	19	5
Reserve Fund	34	18	6	National War Bonds £200 5 per Cent., 1st April, 1928, at cost	200	0	0
Sundry Creditors	138	7	0	Funding Loan 4 per Cent., £2500 at cost ...	2000	0	0
De Morgan Medal Fund	5	18	2	Cash at Bank	204	17	1
Revenue Account balance in hand	77	7	10				
	£3841	3	11		£3841	3	11

REVENUE ACCOUNT for the year 1st November, 1918, to 31st October, 1919.

	Year 1917–18.							Year 1917–18.					
	£	s.	d.	£	s.	d.		£	s.	d.	£	s.	d.
Balance from last year	95	11	0	*144*	*18*	*6*	Printing and Circulating *Proceedings*, &c.	583	19	9	*429*	*9*	*6*
Income tax returned (2 years)	73	6	1		—		Purchases for Library	3	9	6	*5*	*9*	*2*
Dividends on—							Rent	20	0	0	*26*	*0*	*0*
Lord Rayleigh's Fund ...	34	4	0	*35*	*1*	*6*	Postages and Sundries	6	6	0	*5*	*16*	*5*
The Society's Investments	120	2	10	*104*	*11*	*3*	Balance in hand on 31st October, 1919...	77	7	10	*95*	*11*	*6*
Entrance Fees	21	0	0	*3*	*3*	*0*							
140 Annual Subscriptions	147	0	0	*150*	*3*	*0*							
Sale of *Proceedings*, &c.	187	9	2	*118*	*8*	*10*							
Advertisements	12	10	0		—								
	£691	3	1	*556*	*6*	*1*		£691	3	1	*556*	*6*	*1*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.		£	s.	d.
Balance on 31st October, 1918	2	3	11	Balance in hand 31st October, 1919	5	18	2
Dividends received	2	2	0				
Income tax returned (2 years)	1	12	3				
	£5	18	2		£5	18	2

STATEMENT OF TRUST FUNDS, 31st October, 1919.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
Do. do. do. do. £200 Guaranteed 3 per Cent. Stock.

This Fund is held on trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 Five per Cent. Preference Stock.

13th November, 1919.

Signed on behalf of the Council:—
J. E. CAMPBELL,
A. E. WESTERN.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN J. C. CUNNINGHAM,
Lt.-Col. late R.E., Auditor.

24th November, 1919.

uniform functions, it is easily seen that the three following sets can all be connected:

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, 0\right), \quad \left(\frac{\pi}{3}, \frac{\pi}{3}, 0\right), \quad \left(\frac{\pi}{2}, 0, 0\right), \quad \left(\frac{\pi}{3}, 0, 0\right), \quad (0, 0, 0),$$

$$\left(\frac{\pi}{2}, \frac{\pi}{4}, 0\right), \quad \left(\frac{\pi}{4}, \frac{\pi}{4}, 0\right), \quad \left(\frac{\pi}{2}, 0, 0\right), \quad (0, 0, 0),$$

$$\left(\frac{\pi}{2}, \frac{\pi}{6}, 0\right), \quad \left(\frac{\pi}{3}, 0, 0\right), \quad \left(\frac{\pi}{6}, \frac{\pi}{6}, 0\right).$$

The corresponding triangle functions can then all be expressed in terms of that derived from the triangle (0, 0, 0). This is, in fact, reflected in many known results in the theory of Elliptic Modular Functions.

For example, the function of the triangle (0, 0, 0) in its simplest form is given by

$$z = k^2, \quad w = iK'/K,$$

k, K, K' being the Jacobian constants.

The function of the triangle $(0, \pi/2, \pi/3)$ can be deduced by a method of Burnside (*l.c.*) or of Klein.* It is found to be the well known expression

$$\frac{4}{27} \frac{(z^2-z+1)^3}{z^2(z-1)^2}.$$

As the substitutions for which a triangle function remains unchanged are obtained by successive inversions in two sides of the triangle, this method provides a simple means of determining those substitutions.

* *Math. Annalen*, Vol. 12 (1877), p. 171.

DETERMINANTAL SYSTEMS OF CO-APOLAR TRIADS ON A CUBIC CURVE

By WILLIAM P. MILNE, M.A., D.Sc.

[Received May 7th, 1918.—Read May 9th, 1918.]

1. *Introduction.*

In the *Proc. London Math. Soc.*, Ser. 2, Vol. 15, pp. 309–313, I studied in some detail the grouping of the triads of points co-apolar with a given triad on a cubic curve. I found that if ABC be an assigned "apolar triangle," and if P be any point chosen arbitrarily on the cubic curve, then *two* complementary dyads of points Q_2R_3 and Q_3R_2 can be found such that $P_1Q_2R_3$ and $P_1Q_3R_2$ are each co-apolar with ABC. If, thereafter, any other point of those obtained above such as Q_2 be taken, the dyad complementary to Q_2, other than P_1R_3 may be denoted by P_3R_1, and so on, with the other points. Proceeding in this way, it was shewn that a *closed* system of nine points was obtained, namely :

$$\begin{matrix} P_1 & P_2 & P_3, \\ Q_1 & Q_2 & Q_3, \\ R_1 & R_2 & R_3, \end{matrix}$$

such that if any one of the six triads defined by the terms of the expansion of the above determinantal form were taken, each triad so obtained would be co-apolar with ABC. Such a system of nine points was called a "determinantal system." The configuration is in its essence a symmetrical one, but the method of obtaining it in the above paper gave no clue as to the symmetry of its form, nor any convenient means of visualizing its structure as a geometrical unit. The purpose of the present communication is to establish the symmetry of the configuration.

2. *The Direct Proposition.*

Let UVW and XYZ be any two coplanar triangles, and let us consider the pencil of cubic curves circumscribing these two triangles and apolar

to them. This pencil of cubics will have in common three other points ABC, which we shall refer to as the triangle "complementary" to UVW and XYZ. It was shewn in the *Proceedings of the Edinburgh Mathematical Society*, Vol. 30, Session 1911–12, that every member of the pencil of cubics is apolar to the complementary triangle ABC. It will now be proved that *every member of the pencil of cubic curves circumscribing two triangles and apolar to them cuts the nine lines joining the vertices of these triangles in a determinantal system of points co-apolar with the complementary triangle.*

Let a cubic of the pencil cut the straight lines

$$\begin{matrix} UX, & WY, & VZ, \\ VY, & UZ, & WX, \\ WZ, & VX, & UY, \end{matrix}$$

in the respective points,

$$\begin{matrix} P_1 & P_2 & P_3, \\ Q_1 & Q_2 & Q_3, \\ R_1 & R_2 & R_3. \end{matrix}$$

Let us consider first of all the nodal members of the pencil. It was shewn in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243, that the necessary and sufficient condition that two apolar triads on a nodal cubic be co-apolar is that the pencils subtended by them at the node shall be apolar. It was also shewn in the above-mentioned paper in the *Proc. Edin. Math. Soc.*, that the canonical form for three complementary apolar triangles on the nodal cubic $x = 6t^2$, $y = 6t$, $z = -(1+t^3)$ can be taken to be

$$ABC \equiv t^3+3pt^2+3qt+pq = 0, \tag{1}$$

$$XYZ \equiv t^3+3rt^2+3st+rs = 0, \tag{2}$$

$$UVW \equiv pqrst^3+3qst^2+3prt+1 = 0. \tag{3}$$

If now we project the triad XYZ through the point t_0 of the nodal cubic on to the nodal cubic again, so as to obtain the triad

$$rst_0^3t^3-3st_0^2t^2+3rt_0t-1 = 0, \tag{4}$$

we see that the apolar triads (1) and (4) subtend pencils at the node which are apolar to one another, if

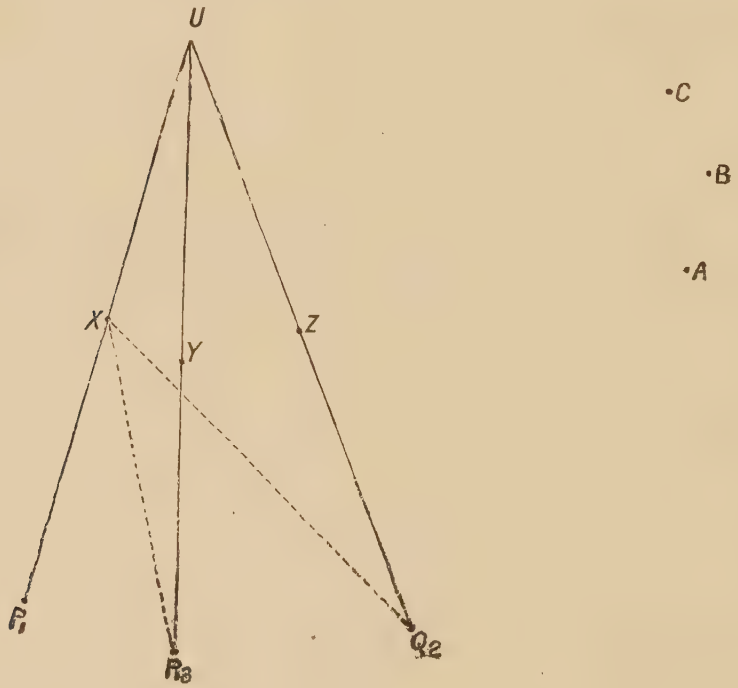
$$pqrst_0^3+3qst_0^2+3prt_0+1 = 0,$$

i.e., if t_0 belong to the triad (3).

Hence we obtain the following result:—

If a nodal cubic circumscribe the two triangles ABC, XYZ and be apolar to them, and if XYZ be projected through any one of the vertices of the complementary triangle UVW on to the cubic again, the apolar triangle thus obtained is co-apolar with ABC.

If U be the vertex chosen through which the projection is to take place, we have thus shewn that the triangle $P_1 Q_2 R_3$ is co-apolar with ABC. Similarly with the other members of the determinantal configuration defined by the intersections of the nodal cubic with the lines joining the vertices of the triangles XYZ, UVW. The required theorem has therefore been proved for the nodal members of the pencil of cubics. We proceed to establish the general result for those members of the pencil which do not possess singular points.



In the above figure, let U be one of the vertices of the triangle complementary to ABC, XYZ. Let R_3 be any point on UY, and let the cubic circumscribing ABC, XYZ and apolar to each of them which passes through R_3, cut UZ in Q_2. Then plainly a one-to-one algebraic correspondence exists between R_3 and Q_2. Again, let Q_2' be taken on UZ, so that $X[U, R_3, Q_2']$ is apolar to $X[ABC]$. It is plain, once more, that a

one-to-one algebraic correspondence exists between R_3 and Q_2'. But it has been shewn above that in the case of the *twelve* nodal cubics of the pencil, Q_2 and Q_2' coincide. Hence Q_2 and Q_2' must coincide for *every* member of the pencil.

We thus have the following result :—

If U be one of the vertices of the triangle complementary to ABC, XYZ, and if a cubic circumscribing ABC, XYZ, and apolar to each of them, cut UX, UY, UZ in $P_1R_3Q_2$ respectively, then $P_1R_3Q_2$ subtends at X a pencil of lines which is apolar to $X[ABC]$. Similarly for Y and Z.

But since U lies on the "apolar locus" of ABC, XYZ, we see that $U[ABC]$ is apolar to $U[XYZ]$, *i.e.* to $U[P_1R_3Q_2]$. Hence the "apolar locus" of ABC, $P_1Q_2R_3$ must pass through the points X, Y, Z, U, and therefore have *ten* points in common with the above cubic circumscribing ABC, XYZ, and apolar to each of them, and passing through $P_1Q_2R_3$. The two cubics must therefore coincide. We see therefore with respect to the cubic of the above pencil under consideration that $P_1Q_2R_3$ is a triangle co-apolar with ABC. The final result is now established :

The lines joining the vertices of any two apolar triangles on a cubic curve cut the curve again in nine points forming a determinantal system, co-apolar with the complementary triangle of the given apolar triangles.

3. *The Converse Proposition.*

We next proceed to establish the converse :

If ABC be a given apolar triad and $|PQR|$ a definite determinantal system of co-apolar triads, then $|PQR|$ can in a singly-infinite number of ways be regarded as the further intersections of the cubic curve with the lines joining the vertices of two apolar triangles XYZ, UVW complementary with ABC.

Let U be any point on the cubic curve and let us join U to the three points P_1, Q_2, R_3, so that UP_1, UR_3, UQ_2 cut the cubic again in X, Y, Z respectively. Then, since $P_1Q_2R_3$ is an apolar triangle, we know that XYZ is an apolar triangle. Further, since $U[XYZ]$ is apolar to $U[ABC]$, we see that U is one of the vertices of the triangle complementary to ABC, XYZ. Let V, W be the other two vertices of the triangle complementary to ABC, XYZ. Let XV, XW meet the curve again in R_2', Q_3'. Then, since $X[UVW]$ is apolar to $X[ABC]$, we see from § 2 that P_1, R_2', Q_3 are co-apolar with ABC. Thus the complementary dyads of P_1 are Q_2, R_3, and Q_3', R_2' (according to the notation of the *Proc. London Math.*

Soc., Ser. 2, Vol. 15, pp. 309–313). Hence Q_3', R_2' must coincide with Q_3, R_2 of the determinantal system $|PQR|$. Proceeding in this way, we see that all the lines joining pairs of vertices of XYZ and UVW must pass each through a member of the determinantal system $|PQR|$. Moreover, we chose U arbitrarily on the cubic curve, and thence obtained the other five points XYZ, VW uniquely. Hence, since U can be chosen in a singly-infinite number of ways, we see that the given determinantal system can be regarded as being deduced from a singly-infinite system of pairs of apolar triangles.

4. *Apolar Triangles on a Conic and Tangents to the Cayleyan.*

The case when the two fundamental apolar triangles XYZ, UVW lie on a conic is of considerable geometrical interest. In this case the complementary triad ABC consists of three collinear points lying on the line joining the poles of the conic (regarded as a polar conic) with respect to the triangles XYZ, UVW (regarded as cubic curves), while the tangents at A, B, C to any cubic of the pencil circumscribing XYZ, UVW and apolar to them are concurrent. Moreover, the triads $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ consist each of three collinear points, as do also the triads $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$. In fact, we obtain the determinantal system dealt with in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243, in which the given cubic is shewn to pass through the nine intersections of three concurrent lines with three other concurrent lines, it being further shewn that the points of concurrency are "corresponding points" on the Hessian curve. Hence in the above configuration, $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ are straight lines passing through one Hessian point of the collinear triad ABC, while $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$ pass through the other.

We may proceed further in the search for geometrical properties. It is known that an apolar triad becomes a tangent to the Cayleyan if the three points constituting the triad are collinear. Hence we may take XYZ, UVW to be any two tangents m, n to the Cayleyan. The complementary triad ABC will lie on a line l which is also a tangent to the Cayleyan. Given m and n, it is easy to construct l. For, it was shewn in the *Proc. Edin. Math. Soc.*, in the paper quoted above, that the conic $XYZUVW$ of last paragraph harmonically separates the Hessian points of the triad ABC. Hence, the conic composed of l, m must harmonically separate the Hessian points of UVW, *i.e.* the line l is the join of the harmonic conjugates of the point mn with respect to the Hessian points of XYZ and UVW. We thus see that the lines joining the points of the

collinear triads lying on the lines m and n cut the cubic again in nine points which lie three by three on six straight lines that pass through one or other of two "corresponding points" on the Hessian of the given cubic curve, whose position is defined on the line l in the way stated above.

If m and n together constitute a polar-conic of the given cubic, and m', n' be the polar-conic of the point mn, it is easy to shew that the line l is in this case that line which with the join of mn and $m'n'$ constitutes a polar-conic of the original cubic, since every polar-conic cuts the join of a pair of "corresponding points" harmonically. We thus obtain the following theorem:—

If a degenerate polar-conic of the original cubic curve consist of the lines m and n, and if m', n' be the polar conic of the point mn, the nine lines joining the points of section of m and n with the cubic cut the curve again in nine points which lie three by three on six lines passing through one or other of the pair of "corresponding points" in which the line l cuts the Hessian curve, where l is the line which together with the join of mn and m'n' constitutes a degenerate polar-conic of the original cubic.

Symmetry shews *that the joins of the points of section of m', n' with the cubic curve pass through one or other of the same pair of "corresponding points" on the Hessian curve.*

Many other theorems of a like nature might be stated by taking particular positions for two tangents to the Cayleyan.

I take the opportunity of expressing my indebtedness to Dr. William L. Marr, of Aberdeen, a communication from whom first suggested the mode of treatment in the present paper.

THE ELECTROMAGNETIC PROPERTIES OF COILS OF CERTAIN FORMS

By A. YOUNG.

[Read May 9th, 1918.—Received September 20th, 1918.]

THE magnetic potential of a closed circuit at any point outside the circuit is measured by ωi, where ω is the solid angle subtended by the circuit at the point, and i is the strength of the current. The magnetic potential of a coil can be obtained by summing the expressions ωi for each turn.

The induction parallel to the axis of a coil at any point is, of course, the ratio of the difference between the potentials at two neighbouring points—such that the straight line joining them is parallel to the axis—to the distance between the points. The effect would be the same if we took the difference between the potentials of two coils at the same point, the one coil being obtained from the other by a small displacement parallel to the axis. Thus, for a single layer coil of n turns to the centimetre, $n\Omega i$ is the magnetic induction parallel to the axis at a point, where Ω is the solid angle subtended by the coil itself at the point.

In what follows coils are considered in which each turn is in the form of a rectangle. The solid angle subtended by a rectangle at any point may be represented as the inverse tangent of a certain ratio. Hence is deduced by integration the magnetic potential of such a coil. The coil is looked upon as an infinitely thin current sheet.

A further double integration enables either the self induction of the coil to be written down, or else the mutual induction of two such coils which are such that the three principal planes are parallel; *i.e.* the planes at right angles to the axis and those through the axis at right angles to the sides of the rectangular turns.

In § 3 the potential is obtained for a coil of the same form but of finite thickness, and thence by double integration the mutual induction of two such coils whose axes and sides are parallel.

In § 4 the self induction of this coil is given.

In § 5 the question of the looseness of the windings is taken into

account, and in the remaining paragraphs the discussion of § 5 is applied to the cases considered.

1. Firstly, consider a coil in which each turn is of the shape of a rectangle whose sides are $2a$, $2b$; the turns being closely wound, so that we may consider the coil as equivalent to an infinitely thin current sheet.

Take the axis of the coil for axis z, its middle point for origin, and the axes of x and y parallel to the sides $2a$, $2b$ respectively of the coil. The axes are so chosen that they form a right handed system. Now the solid angle subtended by a rectangle $ABCD$ at any point O on the straight line through A at right angles to its plane is

$$\tan^{-1}\frac{AB\,.\,AD}{OA\,.\,OC}.$$

The potential of the turn in the plane $z=\xi$ (supposed to carry unit current in the positive direction from x to y round z) at the point (x, y, z) is

$$\tan^{-1}\frac{(a+x)(b+y)}{(z-\xi)\sqrt{\{(a+x)^2+(b+y)^2+(z-\xi)^2\}}}$$

$$+\tan^{-1}\frac{(a-x)(b+y)}{(z-\xi)\sqrt{\{(a-x)^2+(b+y)^2+(z-\xi)^2\}}}$$

$$+\tan^{-1}\frac{(a+x)(b-y)}{(z-\xi)\sqrt{\{(a+x)^2+(b-y)^2+(z-\xi)^2\}}}$$

$$+\tan^{-1}\frac{(a-x)(b-y)}{(z-\xi)\sqrt{\{(a-x)^2+(b-y)^2+(z-\xi)^2\}}}.$$

Now, if $$r^2=x^2+y^2+z^2,$$

$$\int\tan^{-1}\frac{xy}{zr}\,dz=z\tan^{-1}\frac{xy}{zr}-x\tanh^{-1}\frac{y}{r}-y\tanh^{-1}\frac{x}{r}.$$

Then the potential of the coil at (x, y, z) is

$$Vi=ni\left[\left[\left[\xi\tan^{-1}\frac{\xi\eta}{\xi r}-\xi\tanh^{-1}\frac{\eta}{r}-\eta\tanh^{-1}\frac{\xi}{r}\right]_{\xi=x-a}^{x+a}\right]_{\eta=y-b}^{y+b}\right]_{\zeta=z-h}^{z+h},$$

where $r^2=\xi^2+\eta^2+\zeta^2$, n is the number of terms per unit length, $2h$ is the length of the coil, i is strength of the current: and the values of the variables inserted above and below the end brackets imply that we have to take the difference of the values of the functions when the variables have these values.

2. The magnetic induction parallel to the axis of the coil is $-\partial V/\partial z$. The mutual inductance of this coil and another—whose axis is parallel to its axis, whose turns are rectangles having sides parallel to the axes of x and y and of lengths $2a'$, $2b'$, whose length is $2h'$, whose centre is (α, β, γ) and which has n' turns per unit length—is

$$n' \int_{\alpha-a'}^{\alpha+a'} \int_{\beta-b'}^{\beta+b'} \int_{\gamma-h'}^{\gamma+h'} -\frac{\partial V}{\partial z}\, dx\, dy\, dz.$$

We require then the indefinite integral

$$\begin{aligned} \iint \Big[-\xi \tan^{-1}\frac{\xi\eta}{\xi r} + \xi \tanh^{-1}\frac{\eta}{r} + \eta \tanh^{-1}\frac{\xi}{r} \Big] d\xi\, d\eta \\ = -\xi\eta\zeta \tan^{-1}(\xi\eta/\zeta r) + \tfrac{1}{2}\eta(\xi^2-\zeta^2)\tanh^{-1}(\eta/r) + \tfrac{1}{2}\xi(\eta^2-\zeta^2)\tanh^{-1}(\xi/r) \\ + (2\zeta^2-\xi^2-\eta^2)\, r/6 \\ = \phi(\xi, \eta, \zeta). \end{aligned}$$

Thus the mutual inductance of the two coils is

$$nn' \Big[\Big[\Big[\Big[\Big[\Big[\phi(\xi, \eta, \zeta) \Big]_{\xi=x-a}^{x+a} \Big]_{\eta=y-b}^{y+b} \Big]_{\zeta=z-h}^{z+h} \Big]_{x=\alpha-a'}^{\alpha+a'} \Big]_{y=\beta-b'}^{\beta+b'} \Big]_{z=\gamma-h'}^{\gamma+h'}.$$

Noticing that ϕ is unaltered by the change of sign of either of its three arguments, we can deduce the self inductance of the original coil in the form

$$8n^2 \Big[\Big[\Big[\phi(\xi, \eta, \zeta) \Big]_{\xi=0}^{2a} \Big]_{\eta=0}^{2b} \Big]_{\zeta=0}^{2h}.$$

If the second coil have its axis parallel to the axis y, and the sides of its rectangular turns parallel to the axes of x and z respectively, the mutual inductance becomes

$$\begin{aligned} nn' \Big[\Big[\Big[\Big[\Big[\Big[-\tfrac{1}{2}\xi\zeta^2 \tan^{-1}\frac{\xi\eta}{\zeta r} - \tfrac{1}{2}\xi\eta^2 \tan^{-1}\frac{\xi\zeta}{\eta r} - \tfrac{1}{6}\xi^3 \tan^{-1}\frac{\eta\zeta}{\xi r} + \xi\eta\zeta \tanh^{-1}\frac{\xi}{r} \\ + \frac{3\xi^2\zeta-\zeta^3}{6} \tanh^{-1}\frac{\eta}{r} + \frac{3\xi^2\eta-\eta^3}{6} \tanh^{-1}\frac{\zeta}{r} \\ - \tfrac{1}{3}\eta\zeta r \Big]_{\xi=x-a}^{x+a} \Big]_{\eta=y-b}^{y+b} \Big]_{\zeta=z-h}^{z+h} \Big]_{x=\alpha-a'}^{\alpha+a'} \Big]_{y=\beta-h'}^{\beta+h'} \Big]_{z=\gamma-c'}^{\gamma+c'}. \end{aligned}$$

3. Let us now suppose that the coil is equivalent to a current sheet not infinitely thin, but such that the section of it by a plane perpendicular to the axis is bounded by two rectangles with parallel sides of lengths $2a$. $2b$, and $2a+2t$, $2b+2t$ respectively.

To find the potential at (x, y, z) we require the integral

$$\int\left[\xi\tan^{-1}\frac{(\xi+t)(\eta+t)}{\xi r}-(\xi+t)\tanh^{-1}\frac{\eta+t}{r}-(\eta+t)\tanh^{-1}\frac{\xi+t}{r}\right]dt$$

$$=\tfrac{1}{2}\pi\xi t+\xi(\xi+t)\tan^{-1}\left[\{(2t+\xi+\eta)(t+\eta)-r^2\}/\xi r\right]$$
$$+\xi(\eta+t)\tan^{-1}\left[\{(2t+\xi+\eta)(t+\xi)-r^2\}/\xi r\right]$$
$$+\tfrac{1}{2}\left[\xi^2-(t+\xi)^2\right]\tanh^{-1}\{(t+\eta)/r\}+\tfrac{1}{2}\left[\xi^2-(t+\eta)^2\right]\tanh^{-1}\{(t+\xi)/r\}$$
$$-(1/2\sqrt{2})\{2\xi^2-(\xi-\eta)^2\}\tanh^{-1}\{(2t+\xi+\eta)/r\sqrt{2}\},$$

where $$r^2=(\xi+t)^2+(\eta+t)^2+\xi^2.$$

Let us call this function V_t; then the potential of the coil at (x, y, z) is the sum of the four values of

$$ni\left[V_t-V_0\right]_{\zeta=z-h}^{z+h},$$

obtained by giving to ξ the values $a\pm x$, and to η the values $b\pm y$ simultaneously. Here n is the number of turns per unit area of the cross section through the axis at right angles to one of the sides of the coil. In using the function V_t it will be observed that the term $\frac{1}{2}\pi\zeta t$ only adds to the potential a term independent of (x, y, z), and therefore can be neglected.

In order to find inductances in this case, we want to integrate with respect to X and Y the function

$$V'=ZX\tan^{-1}\left[\{(X+Y)Y-R^2\}/ZR\right]+ZY\tan^{-1}\left[\{(X+Y)X-R^2\}/ZR\right]$$
$$-(1/2\sqrt{2})\{2Z^2-(X-Y)^2\}\tanh^{-1}\{(X+Y)/R\sqrt{2}\}$$
$$+\tfrac{1}{2}(Z^2-X^2)\tanh^{-1}(Y/R)+\tfrac{1}{2}(Z^2-Y^2)\tanh^{-1}(X/R),$$

where $$R^2=X^2+Y^2+Z^2.$$

We find

$$6\int V'dX=(3X^2-Z^2)Z\tan^{-1}\left[\{(X+Y)Y-R^2\}/ZR\right]$$
$$-(3Y^2-6XY-Z^2)Z\tan^{-1}\left[\{(X+Y)X-R^2\}/ZR\right]$$
$$-(3/\sqrt{2})\{6Z^2(X-Y)-(X-Y)^3\}\tanh^{-1}\{(X+Y)/R\sqrt{2}\}+(Y^2-Z^2)R$$
$$+(3Z^2-X^2)X\tanh^{-1}(Y/R)+(Y^3-3XY^2+3XZ^2-3YZ^2)\tanh^{-1}(X/R),$$

and $$24\iint V'dXdY$$

$$= 4Z\{(3Y-X)X^2+(X-Y)Z^2\}\tan^{-1}[\{(X+Y)Y-R^2\}/ZR]$$

$$+4Z\{(3X-Y)Y^2+(Y-X)Z^2\}\tan^{-1}[\{(X+Y)X-R^2\}/ZR]$$

$$-(1/\sqrt{2})\{4Z^4-12Z^2(X-Y)^2+(X-Y)^4\}\tanh^{-1}\{(X+Y)/R\sqrt{2}\}$$

$$+\{X^3+Y^3-2(X+Y)Z^2\}R$$

$$+\{2Z^4+12Z^2XY-6Z^2Y^2+Y^4-4XY^3\}\tanh^{-1}(X/R)$$

$$+\{2Z^4+12Z^2XY-6Z^2X^2+X^4-4YX^3\}\tanh^{-1}(Y/R)$$

$$= \psi(X, Y, Z).$$

For the mutual inductance of this coil and a second which is equivalent to a thin rectangular current sheet, the axes of the coils being parallel and also their plane boundaries, we have

$$nn'\Big[\Big[\Big[\Big[\Big[\Big[\Big[\psi(X, Y, Z)\Big]_{X=t+a-\xi}^{t+a+\xi}\Big]_{Y=t+b-\eta}^{t+b+\eta}\Big]_{Z=\zeta-h}^{\zeta+h}\Big]_{\xi=a-a'}^{a+a'}\Big]_{\eta=\beta-b'}^{\beta+b'}\Big]_{\zeta=\gamma-h'}^{\gamma+h'}\Big]_{t=0}^{t}:$$

where (α, β, γ) is the centre; $2a'$, $2b'$ the lengths of the sides of a turn; $2h'$ the axial length; and n' the number of turns per unit axial length of the second coil.

4. The self inductance of the coil considered is a sum of two parts of which the first is

$$L_1 = 2n^2t\Big[\Big[\Big[\Big[\Big[\Big[\psi(X, Y, Z)\Big]_{X=t+a-\xi}^{t+a+\xi}\Big]_{Y=t+b-\eta}^{t+b+\eta}\Big]_{\xi=-a}^{a}\Big]_{\eta=-b}^{b}\Big]_{t=0}^{t}\Big]_{Z=0}^{h}.$$

The second part L_2 is that due to the lines of induction which cut the conducting material.

This is a sum of terms of which the typical one is obtained from the linkage of the flux due to the current in that part of the coil which is bounded by the rectangles whose sides are $2a+2t$, $2b+2t$, and $2a+2\tau$. $2b+2\tau$, with the lines of induction which pass through the area bounded by the rectangles whose sides are $2a+2\tau$, $2b+2\tau$, and $2a+2\tau-2\delta\tau$. $2b+2\tau-2\delta\tau$, the rectangles being symmetrically situated with respect to

the coil, and in the same section perpendicular to the axis; and τ varying from 0 to t.

The result after integration is

$$L_2 = 8n^2\Big[\Big[-F_1(\vartheta, -2t-2a, -2t-2b, z)+F_1(\vartheta, -t-2a, -t-2b, z)$$
$$-F_1(\vartheta, 0, 0, z)+F_1(\vartheta, -t, -t, z)+F_2(\vartheta, -2t-2b, z)$$
$$+F_2(\vartheta, -2t-2a, z)+F_2(\vartheta, 2b, z)+F_2(\vartheta, 2a, z)$$
$$-tF_3(\vartheta, 2b, z)-tF_3(\vartheta, 2a, z)\Big]_{\vartheta=0}\Big]_{z=0}^{2h},$$

where $F_1(\vartheta, a, \beta, z)$

$$=-z\{\tfrac{1}{4}\vartheta^4+\tfrac{1}{3}(a+\beta)\vartheta^3+\tfrac{1}{2}a\beta\vartheta^2\}\tan^{-1}\frac{zr}{(\vartheta+a)(\vartheta+\beta)}$$

$$+(1/60)\{5a^3(a-2\beta)z-10a(a-\beta)z^3+z^5\}\tan^{-1}[\{(2\vartheta+a+\beta)(\vartheta+\beta)-r^2\}/zr]$$

$$+(1/60)\{5\beta^3(\beta-2a)z-10\beta(\beta-a)z^3+z^5\}\tan^{-1}[\{(2\vartheta+a+\beta)(\vartheta+a)-r^2\}/zr]$$

$$-(1/120)[12(\vartheta+a)^5+15(\beta-2a)(\vartheta+a)^4+20a(a-\beta)(\vartheta+a)^3$$
$$-z^2\{20(\vartheta+a)^3+30(\beta-2a)(\vartheta+a)^2+60a(a-\beta)(\vartheta+a)\}$$
$$-5z^4(\beta-2a)]\tanh^{-1}\{(\vartheta+\beta)/r\}$$

$$-(1/120)[12(\vartheta+\beta)^5+15(a-2\beta)(\vartheta+\beta)^4+20\beta(\beta-a)(\vartheta+\beta)^3$$
$$-z^2\{20(\vartheta+\beta)^3+30(a-2\beta)(\vartheta+\beta)^2+60\beta(\beta-a)(\vartheta+\beta)\}$$
$$-5z^4(a-2\beta)]\tanh^{-1}\{(\vartheta+a)/r\}$$

$$+(1/240\sqrt{2})(a+\beta)\{5(a-\beta)^4-60z^2(a-\beta)^2+16z^4\}\tanh^{-1}\{(2\vartheta+a+\beta)/r\sqrt{2}\}$$

$$+r^5/60-(a+\beta)r^3\vartheta/48-(a^2+\beta^2+3a\beta+13z^2)r^3/120$$

$$-(a+\beta)\{(a-\beta)^2-7z^2\}r\vartheta/48$$

$$-[5(a^2+\beta^2)(a-\beta)^2-z^2\{44(a^2+\beta^2)-18a\beta\}-14z^4]r/240,$$

where $r^2 \equiv (\vartheta+a)^2+(\vartheta+\beta)^2+z^2.$

$$F_2(\mathfrak{I}, \beta, z)$$

$$= (1/6)\{\mathfrak{I}^3(3\mathfrak{I}+2\beta)z-\mathfrak{I}^2z^3\}\tan^{-1}[\{\beta(\mathfrak{I}+\beta)-r^2\}/zr]$$
$$+(1/6)\{3\mathfrak{I}^2(\mathfrak{I}+\beta)^2z-\mathfrak{I}^2z^3\}\tan^{-1}\{(\beta\mathfrak{I}+r^2)/zr\}$$
$$+(1/240\sqrt{2})\{-3(8\mathfrak{I}-\beta)(2\mathfrak{I}+\beta)^4+60(4\mathfrak{I}-\beta)(2\mathfrak{I}+\beta)^2z^2$$
$$-260\beta z^4\}\tanh^{-1}(\beta/r\sqrt{2})$$
$$+(1/24)\{(4\mathfrak{I}+3\beta)\mathfrak{I}^4-6(2\mathfrak{I}+\beta)\mathfrak{I}^2z^2-\beta z^4\}\tanh^{-1}\{(\mathfrak{I}+\beta)/r\}$$
$$-(1/6)(\mathfrak{I}+\beta)\mathfrak{I}^2\{(\mathfrak{I}+\beta)^2-3z^2\}\tanh^{-1}(\mathfrak{I}/r)$$
$$+(1/240)\{-5\beta^4z+20\beta^2z^3+188z^5\}\tan^{-1}\{\beta(2\mathfrak{I}+\beta)/2zr\}$$
$$-(1/120\sqrt{2})\{\beta^5+40\beta^3z^2+140\beta z^4\}\tanh^{-1}\{(2\mathfrak{I}+\beta)/r\sqrt{2}\}$$
$$+(1/720)\{189^3+22\beta\mathfrak{I}^2+28\beta^2\mathfrak{I}-13\beta^3+306\mathfrak{I}z^2-262\beta z^2\}\beta r,$$

where $$r^2 \equiv \mathfrak{I}^2+(\mathfrak{I}+\beta)^2+z^2.$$

$$F_3(\mathfrak{I}, \beta, z)$$

$$= (1/6)\{4\mathfrak{I}^3+3\beta\mathfrak{I}^2-(2\mathfrak{I}+\beta)z^2\}z\tan^{-1}[\{\beta(\mathfrak{I}+\beta)-r^2\}/zr]$$
$$+(1/6)\{(\mathfrak{I}+\beta)^2(4\mathfrak{I}+\beta)-(2\mathfrak{I}+\beta)z^2\}z\tan^{-1}\{(\beta\mathfrak{I}+r^2)/zr\}$$
$$-(1/24\sqrt{2})\{-3(2\mathfrak{I}+\beta)^4+36(2\mathfrak{I}+\beta)^2z^2+52z^4\}\tanh^{-1}(\beta/r\sqrt{2})$$
$$+(1/24)\{5\mathfrak{I}^4+4\beta\mathfrak{I}^3-6\mathfrak{I}(3\mathfrak{I}+2\beta)z^2+z^4\}\tanh^{-1}\{(\mathfrak{I}+\beta)/r\}$$
$$+(1/24)\{-(\mathfrak{I}+\beta)^3(5\mathfrak{I}+\beta)+6(\mathfrak{I}+\beta)(3\mathfrak{I}+\beta)z^2-z^4\}\tanh^{-1}(\mathfrak{I}/r)$$
$$+(1/72)\{\mathfrak{I}^2+\beta\mathfrak{I}+5\beta^2+83z^2\}\beta r,$$

where $$r^2 \equiv \mathfrak{I}^2+(\mathfrak{I}+\beta)^2+z^2.$$

5. So far the current has been supposed to be uniformly distributed in the space occupied by the coil. In practice this space is partly occupied by insulation. A closer approximation to the truth, and one which will exhibit the degree of error introduced by the assumption made, may be obtained in the following manner.

Let $\phi(z)$ be a function of z which with all its derivatives is finite and continuous throughout the region considered.

Consider the sum

$$S = \phi(z+h)-\phi(z+h-k)+\phi(z+h-k-k')-\phi(z+h-2k-k')$$
$$+\phi(z+h-2k-2k')-\phi(z+h-3k-2k')+\ldots-\phi(z-h),$$

where $$2h = \nu k+(\nu-1)k'.$$

Using Taylor's theorem we may write

$$\phi(z+h)-\phi(z+h-k)=k\phi'(z+h)-(k^2/2!)\,\phi''(z+h)+(k^3/3!)\,\phi'''(z+h)-\dots;$$

and therefore

$$\begin{aligned} S = k\{&\phi'(z+h)+\phi'(z+h-k-k')+\phi'(z+h-2k-2k')+\dots \\ &\qquad +\phi'(z-h+k)\} \\ &-(k^2/2!)\{\phi''(z+h)+\phi''(z+h-k-k')+\dots\}+\dots \qquad \text{(I)} \\ = \;& \frac{k}{k+k'}\{\phi(z+h)-\phi(z-h)\} \\ &+\frac{kk'}{2(k+k')}\{\phi'(z+h)-\phi'(z-h)\} \\ &-\frac{kk'(k-k')}{12(k+k')}\{\phi''(z+h)-\phi''(z-h)\} \\ &-\frac{k^2k'^2}{24(k+k')}\{\phi'''(z+h)-\phi'''(z-h)\}+\dots. \qquad \text{(II)} \end{aligned}$$

Let the coefficient of $\phi^{(m)}(z+h)-\phi^{(m)}(z-h)$ in the series (II) be

$$(-)^m(k+k')^m F_m/m!.$$

To calculate F_m we expand each term of (II) thus

$$\phi^{(m)}(z+h)-\phi^{(m)}(z-h)$$

$$=\sum_{r=1}^{\infty}\frac{(k+k')^r}{r!}(-)^{r+1}[\phi^{(m+r)}(z+h)+\phi^{(m+r)}(z+h-k-k')+\dots],$$

and then compare the result with (I), equating the coefficients of

$$[\phi^{(n)}(z+h)+\phi^{(n)}(z+h-k-k')+\dots].$$

Hence

$$\left(\frac{k}{k+k'}\right)^n=\sum_{m=0}^{n-1}\binom{n}{m}F_m.$$

The solution of this set of difference equations is

$$F_m=\frac{1}{m+1}\left\{\left(\frac{k}{k+k'}\right)^{m+1}-\frac{m+1}{2}\left(\frac{k}{k+k'}\right)^m+B_1\binom{m+1}{2}\left(\frac{k}{k+k'}\right)^{m-1}\right.$$

$$\left.-B_3\binom{m+1}{4}\left(\frac{k}{k+k'}\right)^{m-3}+\dots\right\},$$

where B_1, B_3, &c., are Bernoulli's numbers.

6. Let us now apply this to the potential of the coil considered in § 1. Instead of looking on the coil as a single thin current sheet, we will look upon it as made up of a series of thin current sheets distributed in bands of breadth k, separated by bands of insulation of breadth k', thus approximating to the case of a coil wound with a single layer of very thin conducting tape.

The potential function

$$\phi(z) = ni\left[\left[z\tan^{-1}\frac{\xi\eta}{zr} - \xi\tanh^{-1}\frac{\eta}{r} - \tanh^{-1}\frac{\xi}{r}\right]_{\xi=x-a}^{x+a}\right]_{\eta=y-b}^{y+b},$$

considered as a function of z and all its derivatives, is finite and continuous except for points where $z = 0$, and either $x = \pm a$ or $y = \pm b$.

Observing that

$$\phi''(z) = -ni\left[\left[\frac{\xi\eta}{r}\left(\frac{1}{z^2+\xi^2}+\frac{1}{z^2+\eta^2}\right)\right]_{\xi=x-a}^{x+a}\right]_{\eta=y-b}^{y+b},$$

we see that $\phi(z+k)$ can be expanded in ascending powers of k in a convergent series whenever the four inequalities

$$z^2+(x\pm a)^2 > k^2, \qquad z^2+(y\pm b)^2 > k^2$$

are all satisfied.

In obtaining the expression for S given in the last paragraph, Taylor's expansions for

$$\phi\{z+h-k-r(k+k')+k\} \text{ in ascending powers of } k,$$

and for

$$\phi\{z+h-k-k'-r(k+k')+k+k'\} \text{ in ascending powers of } k+k',$$

are used; where r is a positive integer less than $2h/(k+k')$.

We see then that we may use the expression found for S, provided the distance of the point (x, y, z) from the current sheet is greater than $k+k'$. Thus if this condition is satisfied

$$\left[\phi(\zeta)+\tfrac{1}{2}k'\phi'(\zeta)-\tfrac{1}{12}k'(k-k')\,\phi''(\zeta)-\tfrac{1}{24}kk'^2\phi'''(\zeta)+\ldots\right]_{\zeta=z-h}^{z+h}$$

will be a closer approximation to the true potential.

When this condition is not satisfied we may proceed thus: consider a point (x, y, z) near the top boundary of the coil, but such that its distance from the second layer is everywhere greater than $k+k'$

Then we may take the expression

$$\left[\phi(\zeta)+\tfrac{1}{2}k'\phi'(\zeta)\ldots\right]_{\zeta=z-h}^{z+h-k-k'}$$

as the potential of the coil when the top layer is removed. The potential of the top layer by itself is

$$\phi(z+h)-\phi(z+h-k).$$

The potential of the whole coil at this point is the sum of these two expressions.

The case where the point (x, y, z) is near any other part of the coil may be treated in a similar manner.

7. The mutual inductance of two coils whose axes and plane sides are parallel may be treated in the same way. The difficulty arising from divergent series and infinities may be avoided too, for the case when one coil partly encloses the other, by supposing the magnetic force due to the *outer* coil linked with the current in the *inner* coil, and taking the summations in the order due to this arrangement.

The result is given by the expression found in § 2, with the function

$$\left[1+\tfrac{1}{2}k'\frac{\partial}{\partial\xi}-\tfrac{1}{12}k'(k-k')\frac{\partial^2}{\partial\xi^2}+\ldots\right]$$

$$\times\left[1+\tfrac{1}{2}k_1'\frac{\partial}{\partial\xi}-\tfrac{1}{12}k_1'(k_1-k_1')\frac{\partial^2}{\partial\xi^2}+\ldots\right]\phi(\xi,\eta,\zeta),$$

substituted for $\phi(\xi, \eta, \zeta)$: where k_1, k_1' are the breadths of the bands of conducting tape and insulation respectively of the second coil.

To find the self inductance of a coil, we must take the magnetic force due to the whole coil and sum the linkages between the limits for ξ and η: $a-l$, $l-a$, and $b-l$, $l-b$ respectively. This will give us an expression

$$8n^2\left[\left[\left[\left\{1+\tfrac{1}{2}k'\frac{\partial}{\partial\xi}-\tfrac{1}{12}k'(k-k')\frac{\partial^2}{\partial\xi^2}+\ldots\right\}^2\phi(\xi,\eta,\zeta)\right]_{\xi=l}^{2a-l}\right]_{\eta=l}^{2b-l}\right]_{\zeta=0}^{2h}, \quad \text{(III)}$$

where l must be greater than $k+k'$.

We may consider the linkages due to that part of the flux which is near the conductor, and has been excluded from the above, separately.

The magnetic force at a point near one side of the coil consists of two parts, that due to the side to which it is near and that due to the rest of the coil. The latter part will be accounted for if the upper limits in (III) for ξ and η be changed to $2a$ and $2b$, this introduces no infinities. The former part varies from zero—when the point lies actually in an insulating strip—up to $2\pi ni\{(k+k')/k\}$, when the point lies just outside a conducting strip; in the former case the magnetic force is not required for the

linkage. Thus this former part will add to the inductance an amount less than

$$16\pi n^2 hl\,(a+b).$$

8. The same considerations may be applied to the case of a thick coil discussed in §§ 3 and 4. By replacing V_t by

$$\left\{1+\tfrac{1}{2}k'\frac{\partial}{\partial\zeta}-\tfrac{1}{12}k'(k-k')\frac{\partial^2}{\partial\zeta^2}+\ldots\right\}V_t,$$

in § 3, we obtain the potential of a coil formed of a single layer of wire of rectangular cross section $t\times k$ with a thickness k' of insulation between, at any point not too near the conductor.

To find the potential when there are several layers each of thickness t_1 separated by insulation thickness t_2, we apply to V_t the additional operator

$$\left\{1+\tfrac{1}{2}t_1\frac{\partial}{\partial t}-\tfrac{1}{12}t_2(t_1-t_2)\frac{\partial^2}{\partial t^2}+\ldots\right\}.$$

The case when (x, y, z) is near the coil, and also the inductance, may be considered as before.

DIFFRACTION OF WAVES BY A WEDGE OF ANY ANGLE

By H. S. CARSLAW.

[Received October 17th, 1916.—In Revised form December 16th, 1918.—
Read February 13th, 1919.*]

THE solution of the problem of diffraction by a wedge of any angle in the two dimensional cases of plane waves and a line-source, and the three dimensional case of a point-source, is due to Macdonald.† The case of a point-source has been treated differently by Bromwich.‡ If I understand him aright, he believes that Sommerfeld's methods are only of use in the case of the half-plane, or wedge of angle 2π.§ As a matter of fact Sommerfeld himself showed how, by his methods, the solution could be obtained for a wedge of angle $n\pi/m$ (m, n being positive integers), a Riemann's surface of n sheets being used, and he pointed out that, when the angle of the wedge is not commensurable with π, a Riemann's surface of an infinite number of sheets can be employed in the same way.‖ Recently Wiegrefe has given a very complete discussion of the diffraction of plane waves by a wedge of any angle by this method, and he has also given the formal solution for a line-source and point-source.¶ In §§ 2, 3 below, a statement of Sommerfeld's method will be found.

In this paper I also show that the three problems solved by Macdonald by his method of deducing the solution in series for the wave equation from the corresponding solution in series for the equation of potential can be treated by the modification of Sommerfeld's work, employed by me in dealing with Green's functions for the Wedge of any

---

* The publication of this paper has been greatly delayed, owing to the loss of certain correspondence in transit between London and Sydney.—T. J. I'A. B. and G. H. H.

† *Electric Waves*, Appendix D (1902); *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915), p. 410; see also Jackson, *Proc. London Math. Soc.*, Ser. 2, Vol. 1 (1904), p. 393.

‡ *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915), p. 450.

§ *Loc. cit.*, p. 451; foot of page.

‖ Schlömilch's *Zeitschrift f. Math. u. Physik*, Vol. 46 (1901), p. 38.

¶ *Ann. d. Physik*, (4), Vol. 39 (1912), p. 449.

Angle in the Conduction of Heat.* In my method, for the wedge of any angle β, a periodic solution of period 2β is required, and no use is made of Riemann's surfaces.

Also I give, by the same method, the solution for the case of plane waves when the direction in which they are travelling is not perpendicular to the edge of the wedge. This problem, for the wedge of any angle,† is here solved for the first time.

THE EQUATION $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\kappa^2 u=0$.

PLANE WAVES.

Solution of Period $2n\pi$.

1. In my former paper‡ the fundamental solution

$$u_0 = e^{i\kappa r\cos(\theta-\theta')},$$

which represents the disturbance due to plane waves coming from the direction θ', is transformed into the integral

$$\frac{1}{2\pi}\int e^{i\kappa r\cos(\alpha-\theta)}\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}}\,d\alpha,$$

over the path (A) of Fig. 1, in the α-plane. These two curves are supposed symmetrical and asymptotic to the lines $\pm\pi+\theta$ in the shaded intervals. These shaded intervals denote parts of the α-plane in which the imaginary part of $\cos(\alpha-\theta)$ is positive. In this transformation θ' is supposed to lie between $-\pi+\theta$ and $\pi+\theta$. Any path consisting of two such parts, beginning in the shaded part at infinity to the right of θ and ending in the shaded part to the left of θ, without cutting the real axis, will give the same integral.

An expression of period $2n\pi$, n being a positive integer, was then obtained, by taking the following integral over the same path (A) in the

---

* *Proc. London Math. Soc.*, Ser. 2, Vol. 8 (1910), p. 365. This paper will be referred to below as (G.F.).

† The case of Oblique Incidence on the Semi-Infinite Plane has been treated by me previously, cf. *Proc. Edinburgh Math. Soc.*, Vol. 19 (1901), p. 71.

‡ *Proc. London Math. Soc.*, Ser. 1, Vol. XXX (1898), p. 121. This paper on "Multiform Solutions" will be referred to below as (M.S.).

α-plane :—

$$u = \frac{1}{2n\pi} \int e^{ikr\cos(\alpha-\theta)} \frac{e^{i\alpha/n}}{e^{i\alpha/n} - e^{i\theta'/n}} d\alpha. \qquad (1)$$

The following properties of the expression in (1) were proved :—

(i) It is a solution of the differential equation.

(ii) It is periodic in θ, of period $2n\pi$.

(iii) It is finite and continuous.

(iv) As $r \to \infty$, $u \to u_0$ when $|\theta - \theta'| < \pi$, and $u \to 0$ when $\pi < |\theta - \theta'| < 2n\pi$.

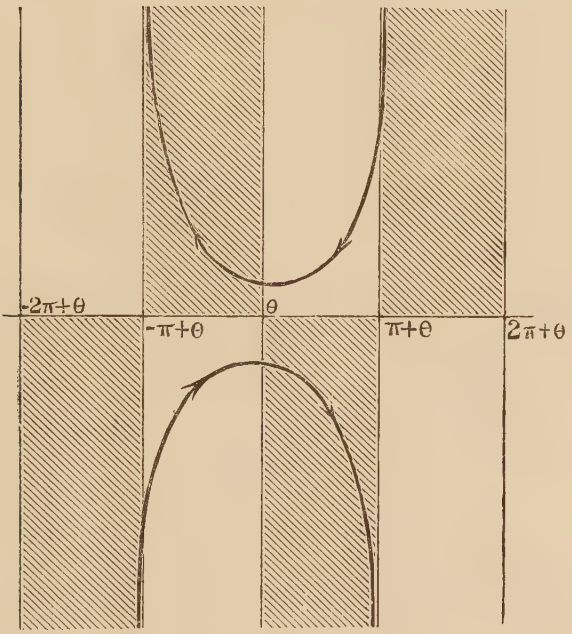


FIG. 1.—The path (A) in the α-plane. Breadth of strip, π.

Now put $\alpha - \theta = \alpha'$ in (1) and we obtain

$$u = \frac{1}{2n\pi} \int e^{ikr\cos\alpha'} \frac{e^{i(\alpha'+\theta)/n}}{e^{i(\alpha'+\theta)/n} - e^{i\theta'/n}} d\alpha',$$

over the path (A') of Fig. 2, in the α'-plane.

In the lower part of the path change the sign of α', and we bring it to

the upper part, and have, finally,

$$u = \frac{1}{2n\pi}\int e^{i\kappa r\cos\alpha'}\left(\frac{e^{i(\alpha'+\theta)/n}}{i(\alpha'+\theta)/n - e^{i\theta'/n}} - \frac{e^{i(\alpha'+\theta)/n}}{e^{i(-\alpha'+\theta)/n} - e^{i\theta'/n}}\right) d\alpha'$$

$$= \frac{i}{2n\pi}\int e^{i\kappa r\cos\alpha'}\frac{\sin\dfrac{\alpha'}{n}}{\cos\dfrac{\alpha'}{n} - \cos\dfrac{\theta-\theta'}{n}}\,d\alpha', \qquad (2)$$

over the *upper* part of the path (A′) in the α'-plane, or any similar path not cutting the real axis, and beginning at infinity in the shaded part to the right of the origin and ending at infinity in the shaded part to the left.

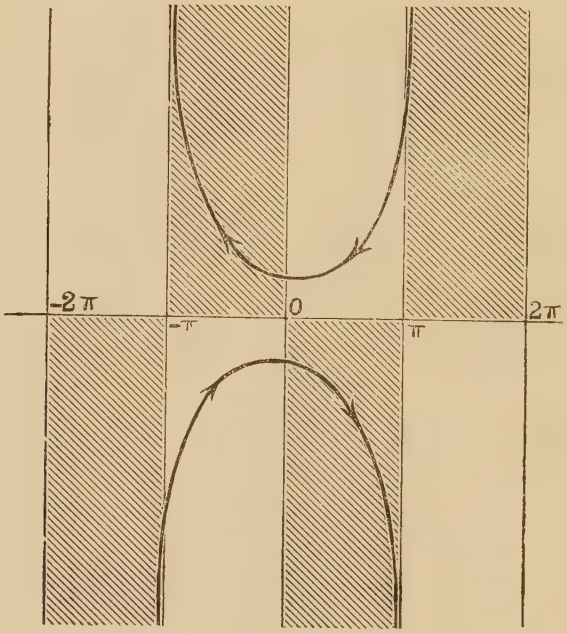


FIG. 2.—Breadth of strip, π; the path (A′) in the α'-plane.

2. *Sommerfeld's solution for the wedge of angle $n\pi/m$ (m and n being positive integers).*

Sommerfeld's method of solution for the case of plane waves and the wedge of any angle $n\pi/m$ (m and n being positive integers) is to form the sums :—

$$U_1 = u(\theta') + u\left(\theta' + 2\frac{n\pi}{m}\right) + u\left(\theta' + 4\frac{n\pi}{m}\right) + \ldots \text{ to } m \text{ terms},$$

and $$U_2 = u(-\theta') + u\left(-\theta' + 2\frac{n\pi}{m}\right) + u\left(-\theta' + 4\frac{n\pi}{m}\right) + \ldots \text{ to } m \text{ terms},$$

denoting the periodic solution of period $2n\pi$ given in § 1 (2) by $u(\theta')$.

The solution of the physical problem is given by

$$u = U_1 \mp U_2, \qquad 0 < \theta < \frac{n\pi}{m}, \qquad 0 < \theta' < \frac{n\pi}{m},$$

the negative or positive sign being chosen according as $u = 0$ or $\partial u/\partial n = 0$ is the boundary condition.

A simple piece of trigonometry reduces this to the form

$$u = \frac{im}{2n\pi} \int e^{ikr\cos\alpha'} \left\{ \frac{\sin\frac{m}{n}\alpha'}{\cos\frac{m}{n}\alpha' - \cos\frac{m}{n}(\theta - \theta')} \mp \frac{\sin\frac{m}{n}\alpha'}{\cos\frac{m}{n}\alpha' - \cos\frac{m}{n}(\theta + \theta')} \right\} d\alpha', \tag{3}$$

over the upper part of the path (A′) of Fig. 2 in the α'-plane.

Sommerfeld speaks of the terms in U_1 and U_2 as the images of $u(\theta')$ in a Riemann's surface of n sheets. He reached the solution for the wedge of any angle by letting n tend to infinity. His method will be described in the next section.*

3. *Sommerfeld's solution for the wedge of any angle β.*

Consider the function defined in § 1 by the equation

$$u = \frac{1}{2n\pi} \int e^{ikr\cos(\alpha - \theta)} \frac{e^{i\alpha/n}}{e^{i\alpha/n} - e^{i\theta'/n}}\, d\alpha,$$

over the path (A) of Fig. 1.

Let n tend to infinity, and we are brought to a solution of the differential equation in the form

$$u = \frac{1}{2i\pi} \int e^{ikr\cos(\alpha - \theta)} \frac{d\alpha}{\alpha - \theta'},$$

over the path (A).

This function vanishes at $r = \infty$, except when θ lies in the interval

---

This method is referred to in various passages of Sommerfeld's papers on this subject. Cf., for example, p. 38 of the paper in Schlömilch's *Zeitschrift*, cited in the introduction. See also Wiegrefe (*loc. cit.*, p. 456).

$|\theta-\theta'| < \pi$, and when θ does lie in this interval we have $u = e^{i\kappa r\cos(\theta-\theta')}$ at $r = \infty$.

As before, this solution reduces to

$$u = \frac{1}{2i\pi}\int e^{i\kappa r\cos\alpha'}\frac{d\alpha'}{\alpha'+\theta-\theta'},$$

over the path (A′) of Fig. 2 in the α'-plane.

And, finally, by changing the sign of α' in the lower part of this path, as in § 1, we reduce the solution to the form

$$u = \frac{1}{2i\pi}\int e^{i\kappa r\cos\alpha'}\left(\frac{1}{\alpha'+\theta-\theta'}+\frac{1}{\alpha'-\theta+\theta'}\right)d\alpha'$$

$$= \frac{1}{2i\pi}\int e^{i\kappa r\cos\alpha'}\frac{2\alpha'}{\alpha'^2-(\theta-\theta')^2}d\alpha', \qquad (4)$$

over the *upper* part only by the path (A′) of Fig. 2.

Denote this solution (4) by $u(\theta')$, remarking that it must not be confused with the previous function of § 2, for which the same symbol was used.

Now add to this the solutions which correspond to waves in directions

$$\theta' \pm 2\beta, \quad \theta' \pm 4\beta, \quad \ldots .$$

Then we have, in place of U_1 of § 2, an infinite series

$$U_1 = \sum_{s=-\infty}^{\infty} u(\theta'+2s\beta)$$

$$= \frac{1}{2i\pi}\int e^{i\kappa r\cos\alpha'}\sum_{-\infty}^{\infty}\frac{2\alpha'}{\alpha'^2-(\theta-\theta'-2s\beta)^2}d\alpha',$$

over the *upper* part only of the path (A′) of Fig. 2.

But we know that

$$\frac{\frac{\pi}{\beta}\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha'-\cos\frac{\pi}{\beta}(\theta-\theta')} = -\sum_{-\infty}^{\infty}\frac{2\alpha'}{\alpha'^2-(\theta-\theta'-2s\beta)^2}.$$

Therefore $$U_1 = \frac{i}{2\beta}\int e^{i\kappa r\cos\alpha'}\frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha'-\cos\frac{\pi}{\beta}(\theta-\theta')}d\alpha', \qquad (5)$$

over the *upper* part only of the path (A′).

Similarly, $U_2 = \sum_{-\infty}^{\infty} u(-\theta' + 2s\beta)$

$$= \frac{1}{2\beta} \int e^{i\kappa r \cos \alpha'} \frac{\sin \frac{\pi}{\beta} \alpha'}{\cos \frac{\pi}{\beta} \alpha' - \cos \frac{\pi}{\beta} (\theta + \theta')} d\alpha,$$

over the *upper* part only of the path (A').

Also the solution of the physical problem is given by

$$u = U_1 \mp U_2, \qquad 0 < \theta < \beta, \qquad 0 < \theta' < \beta,$$

as before.*

4. The solution of period 2β given in (5) can also be obtained, as in §§ 6, 8 and 10 below, by starting from the integral

$$\frac{1}{2\beta} \int e^{i\kappa r \cos (\alpha - \theta)} \frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta} - e^{i\pi\theta'/\beta}} d\alpha, \tag{6}$$

over the path (A) of Fig. 1.

It can be reduced to a series of Bessel's functions as in (G. F.) § 2.

Then we have†

$$U_1 = \frac{\pi}{\beta} \left(J_0(\kappa r) + 2 \sum_1^{\infty} e^{is\pi^2/2\beta} J_{s\pi/\beta}(\kappa r) \cos s\pi(\theta - \theta')/\beta \right), \tag{7}$$

which agrees, when $\beta = \pi$, with the series for $e^{i\kappa r \cos(\theta - \theta')}$, namely

$$J_0(\kappa r) + 2 \sum_1^{\infty} e^{is\pi/2} J_s(\kappa r) \cos s(\theta - \theta').$$

For the case in which $u = 0$ at the surface of the wedge, we have

$$u = \frac{4\pi}{\beta} \sum_1^{\infty} e^{is\pi^2/2\beta} J_{s\pi/\beta}(\kappa r) \sin s\pi\theta/\beta \sin s\pi\theta'/\beta. \tag{8}$$

5. *A line-source at* (r', θ').

We suppose the line-source at (r', θ') parallel to the edge of the wedge.

---

* Cf. Macdonald, *Electric Waves*, p. 192.

† Cf. Sommerfeld, *Math. Ann.*, Vol. 47 (1896), p. 346; if the solutions corresponding to his expression for u at the end of § 5 (3) are taken for θ', $\theta' + 2n\pi/m$, etc., to m terms, and summed, the result agrees with the above when $\beta = n\pi/m$.

In this case we start with the fundamental solution

$$u_0 = K_0(i\kappa R),$$

where $$R^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta').$$

This is transformed into the following integral in the α-plane,

$$\frac{1}{2\pi}\int K_0(i\kappa R)\,\frac{e^{i\alpha}}{e^{i\alpha} - e^{i\theta'}}\,d\alpha,$$

the path being the two symmetrical curves* of Fig. 3. This path we shall again refer to as the path (A).

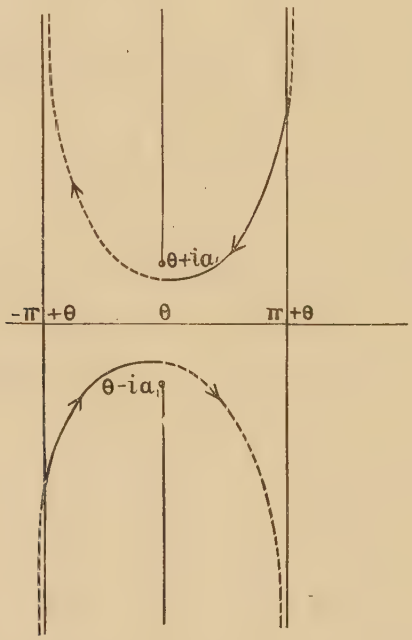


FIG. 3.—Breadth of strip, 2π; the path (A) in the α-plane.

In this integral, R stands for $\sqrt{[r^2 + r'^2 - 2rr'\cos(\alpha - \theta)]}$.

In making this transformation of u_0, we require cuts in the α-plane

---

* In my former paper (M.S.) some modification is needed in Fig. 6. As drawn, it represents the case when R is taken as negative on the real axis.

In this paper I use $K_0(ix)$ for the required Bessel's function of the second kind. With the notation now customary,

$$K_n(ix) = \frac{\pi}{2\sin n\pi}\,e^{-\frac{1}{2}in\pi}\,[J_{-n}(x) - e^{in\pi}J_n(x)].$$

from the branch points of R parallel to the imaginary axis. These branch points are given by

$$\alpha = \theta + 2s\pi \pm i\alpha_1,$$

where

$$\cosh \alpha_1 = \frac{r^2 + r'^2}{2rr'},$$

and s is zero or any positive or negative integer.

The points $\theta \pm i\alpha_1$ are to lie within the path (A) on the right-hand as we pass along this path.

The value of R is fixed in the α-plane, when the cuts are made, if we agree that it is to be positive, say, on the real axis. In this case the dotted parts of the path (A) of Fig. 3 denote points at which the imaginary part of R is negative. In the standard diagram (Fig. 3) the curves are supposed asymptotic to $\pm\pi + \theta$, but the upper curve, for example, could begin at infinity between $\pi + \theta$ and $2\pi + \theta$, and end at infinity between $-\pi + \theta$ and θ.

6. Consider the expression

$$u = \frac{1}{2\beta} \int K_0(i\kappa R) \frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta} - e^{i\pi\theta'/\beta}} d\alpha, \tag{9}$$

over the path (A) of Fig. 3 in the α-plane.

This expression has the following properties :—

(i) It is a solution of the differential equation.

(ii) It is periodic in θ, of period 2β. A change in θ, *e.g.* from θ to ϕ, simply translates the path along the real axis of α, and the term $K_0(i\kappa R)$ is unaltered. Further, if the change in θ is equal to $2s\beta$, where s is any positive or negative integer, the other factor of the integrand is unaltered.

(iii) The choice of the path (A) makes the expression finite and continuous when α_1 is not zero, *i.e.* when r is not equal to r'.

This follows from the approximation for $K_0(ix)$, namely

$$\sqrt{\left(\frac{\pi}{2x}\right)} e^{-i(x+\pi/4)},$$

when $|x|$ is large.*

To learn the behaviour of the expression as $\alpha_1 \to 0$, we note that the

* Cf. Nielsen, *Handbuch d. Th. d. Cylinderfunktionen* (1904), Ch. x.

path (A) may be replaced by rectilinear parts along which the imaginary part of R is negative [cf. (M.S.), pp. 128–9, and 138–9, also (G.F.), p. 367], and circuits round the points $\theta'+2s\beta$, s being zero or any positive or negative integer, lying within $\pm\pi+\theta$. The straight lines give convergent integrals. The circuits round $\theta'+2s\beta$ give for each

$$K_0[i\kappa\sqrt{\{r^2+r'^2-2rr'\cos(\theta-\theta'-2s\beta)\}}],$$

and these terms can be interpreted as images of the source. They are finite, since $r \gtrless r'$.

As $a_1 \to 0$, the same argument holds, but one of the circuits will now give an infinity of the order $K_0(ix)$ when $x \to 0$, if $\theta = \theta'+2s\beta$, s being zero or any positive or negative integer.

Thus, if the range of θ, in the space with which we are concerned, is $0<\theta<\beta$, and if $0<\theta'<\beta$, there will be one point (r', θ'), and only one, at which the expression is infinite.

Also, if we take the solution corresponding to $-\theta'$, where $0<\theta'<\beta$, there will be no infinity in the region $0<\theta<\beta$ for it.

When $r \to \infty$, $u \to 0$, and there are no terms corresponding to waves reflected at infinity. This follows from the argument of (iii) above.

The expression in (9) is reduced as in § 1 to

$$u(\theta') = \frac{i}{2\beta}\int K_0(i\kappa R)\,\frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha'-\cos\frac{\pi}{\beta}(\theta-\theta')}\,d\alpha', \qquad (10)$$

over the *upper* part of the path (A′) of Fig. 4 in the α'-plane, or an equivalent path.

In (10), $$R = \sqrt{(r^2+r'^2-2rr'\cos\alpha')}.$$

It can further be reduced to the series

$$u(\theta') = \frac{\pi}{\beta}\left(J_0(\kappa r)K_0(i\kappa r')+2\sum_1^\infty e^{is\pi^2/2\beta}J_{s\pi/\beta}(\kappa r)K_{s\pi/\beta}(i\kappa r')\cos s\pi(\theta-\theta')\beta\right), \qquad (11)$$

when $r<r'$; and when $r>r'$, we have to interchange r, r' in (11).

7. The physical problem can now be solved as in § 2, by associating with $u(\theta')$ of (10), the corresponding function $u(-\theta')$ for $(r', -\theta')$.

All the conditions imposed upon u, in the region

$$0<r<\infty, \quad 0<\theta<\beta,$$

when (r', θ') is a point in this region, are satisfied by

$$u = u(\theta') \pm u(-\theta'),$$

according as the boundary condition is $\partial u/\partial n = 0$ or $u = 0$.

Thus we have as our solution for the line-source at (r', θ') in the wedge of angle β,

$$u = \frac{i}{2\beta}\int K_0(i\kappa R)\left\{\frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha' - \cos\frac{\pi}{\beta}(\theta-\theta')} \pm \frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha' - \cos\frac{\pi}{\beta}(\theta+\theta')}\right\} d\alpha', \tag{12}$$

the integrals being taken over the *upper* part of the path (A′) of Fig. 4, or an equivalent path.

It will be seen that this agrees with Macdonald's result.*

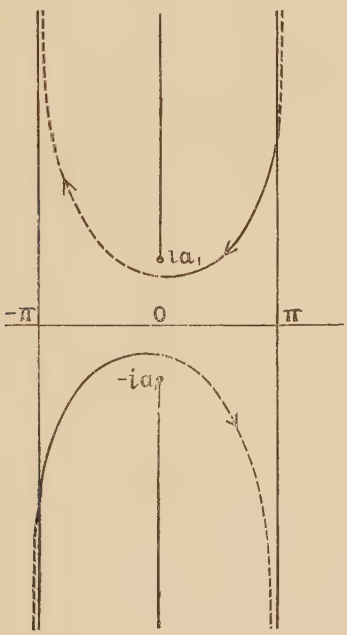


FIG. 4.—Breadth of strip, 2π; the path (A′) in the α'-plane.

* *Electric Waves*, p. 192.

THE EQUATION $\nabla^2 u + \kappa^2 u = 0$.

A point-source at (r', θ', z').

8. For the point-source at (r', θ', z') we start with the fundamental solution

$$u_0 = \frac{e^{-i\kappa R}}{R},$$

where $$R^2 = r^2 + r'^2 + (z-z')^2 - 2rr' \cos(\theta - \theta').$$

Writing $$\cosh a_1 = \frac{r^2 + r'^2 + (z-z')^2}{2rr'},$$

we proceed as in § 5 and transform u_0 into the integral

$$\frac{1}{2\pi} \int \frac{e^{-i\kappa R}}{R} \frac{e^{ia}}{e^{ia} - e^{i\theta'}} da,$$

over the path (A) of Fig. 3 in the a-plane.

In this integral,

$$R = \sqrt{\{r^2 + r'^2 + (z-z')^2 - 2rr' \cos(a-\theta)\}},$$

and the remarks made above as to the cuts in the plane, the fixing of the sign of R, and the interpretation of the dotted parts of the path (A) hold as before.

Finally, the required solution of period 2β is given by

$$u = \frac{1}{2\beta} \int \frac{e^{-i\kappa R}}{R} \frac{e^{i\pi a/\beta}}{e^{i\pi a/\beta} - e^{i\pi\theta'/\beta}} da, \tag{13}$$

over the path (A) of Fig. 3 in the a-plane.

It can be shown, as in § 6, that the expression in (13) has the following properties:—

(i) It is a solution of the differential equation.

(ii) It is periodic in θ, of period 2β.

(iii) It is everywhere finite and continuous, except at the points $(r', \theta' + 2s\beta, z')$, s being zero or any positive or negative integer. At these points it becomes infinite like $e^{-i\kappa x}/x$, when $x \to 0$.

But if the range of θ, in the space with which we are concerned, is $0 < \theta < \beta$, and if $0 < \theta' < \beta$, there will be only one infinity, and that of the proper order in this region, namely, that at (r', θ', z').

Also, if we take the solution corresponding to $-\theta'$, when $0 < \theta' < \beta$, there will be no infinity in the region $0 < \theta < \beta$ for it.

(iv) When $r \to \infty$, $u \to 0$, and there are no terms corresponding waves reflected at infinity.

The expression in (13) can be reduced as before.

We obtain for this solution of period 2β,

$$u(\theta') = \frac{i}{2\beta} \int \frac{e^{-ikR}}{R} \frac{\sin \frac{\pi}{\beta} \alpha'}{\cos \frac{\pi}{\beta} \alpha' - \cos \frac{\pi}{\beta} (\theta - \theta')} d\alpha', \tag{14}$$

where the integral is taken over the *upper* part of the path (A′) of Fig. 4 in the α'-plane, or an equivalent path, and

$$R = \sqrt{\{r^2 + r'^2 + (z - z')^2 - 2rr' \cos \alpha'\}}.$$

9. The physical problem can now be solved as in §§ 2, 7 by associating with $u(\theta')$ of (14), the corresponding function $u(-\theta')$ for $(r', -\theta', z')$.

All the conditions imposed upon u, in the region

$$0 < r < \infty, \quad -\infty < z < \infty, \quad 0 < \theta < \beta,$$

when (r', θ', z') is a point in this region, are satisfied by

$$u = u(\theta') \pm u(-\theta'),$$

according as the boundary condition is $\partial u / \partial n = 0$ or $u = 0$.

Thus we have as our solution for the point-source at (r', θ', z') in the wedge of angle β,

$$u = \frac{i}{2\beta} \int \frac{e^{-ikR}}{R} \left\{ \frac{\sin \frac{\pi}{\beta} \alpha'}{\cos \frac{\pi}{\beta} \alpha' - \cos \frac{\pi}{\beta} (\theta - \theta')} \pm \frac{\sin \frac{\pi}{\beta} \alpha'}{\cos \frac{\pi}{\beta} \alpha' - \cos \frac{\pi}{\beta} (\theta + \theta')} \right\} d\alpha', \tag{15}$$

the integral being taken over the *upper* part of the path (A′) of Fig. 4 in the α'-plane, or an equivalent path.

This agrees with Macdonald's solution.*

10. *Oblique incidence of plane waves on a wedge of angle β.*†

---

* *Loc. cit.*, *Proc. London Math. Soc.*, Ser. 2, Vol. 14 (1915), p. 421.

It will be noticed that I have made some changes in the diagram which corresponds to Fig. 6 of my previous paper (M.S.).

† For oblique incidence on the semi-infinite plane ($\beta = 2\pi$), cf. *Proc. Edinburgh Math. Soc.*, Vol. 19 (1901), p. 71.

The edge of the wedge is supposed not to be perpendicular to the direction of the waves. Spherical polar coordinates (r, θ, ϕ) are used, the origin being in the edge of the wedge, and this line is taken as the initial line for the angle θ or the axis of z.

The space with which we are concerned in the physical problem is given by

$$0 < r < \infty, \quad 0 < \theta < \pi, \quad 0 < \phi < \beta,$$

the plane surfaces of the wedge being $\phi = 0$ and $\phi = \beta$.

We suppose that the line (l, m, n), defined by (θ', ϕ') (Fig. 5) is perpendicular to the front of the waves.

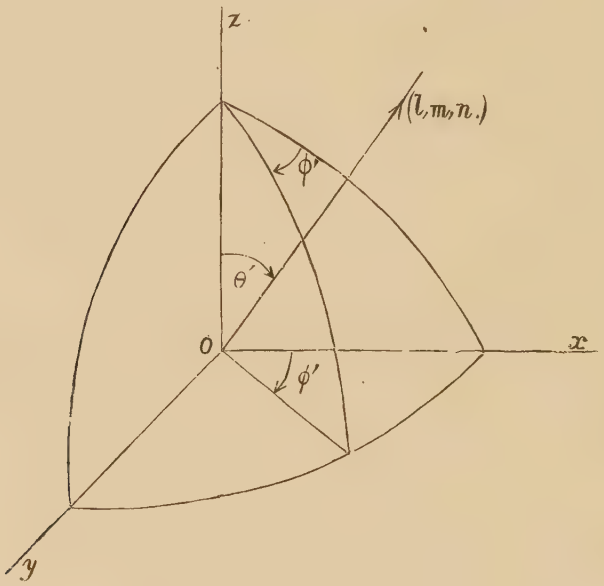


FIG. 5.

The fundamental solution in this case is

$$u_0 = e^{i\kappa(lx+my+nz)} = e^{i\kappa[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')]}.$$

We transform this as in § 1 into the integral

$$\frac{e^{i\kappa r\cos\theta\cos\theta'}}{2\pi} \; e^{i\kappa r\sin\theta\sin\theta'\cos(\alpha-\phi)} \frac{e^{i\alpha}}{e^{i\alpha}-e^{i\phi}}\, d\alpha,$$

over the path (A) of Fig. 1 ; the only change necessary in this figure is to insert ϕ in place of θ.

We are then brought to the solution of period 2β in ϕ,

$$u = \frac{e^{i\kappa r\cos\theta\cos\theta'}}{2\beta}\int e^{i\kappa r\sin\theta\sin\theta'\cos(\alpha-\phi)}\frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta}-e^{i\pi\phi'/\beta}}\,d\alpha, \qquad (16)$$

over the path (A) of Fig. 1 in the α-plane.

This expression has the following properties:—

(i) It is a solution of the differential equation.

(ii) It is periodic in ϕ, of period 2β.

(iii) It is finite and continuous.

(iv) When $r\to\infty$, it takes the value $\Sigma e^{i\kappa r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'-2s\beta)}$, the summation extending to the angles $\phi'+2s\beta$ which lie within $\pm\pi+\phi$, s being zero or any positive or negative integer.

The expression in (16) is reduced, as before, to

$$u(\phi') = \frac{ie^{i\kappa r\cos\theta\cos\theta'}}{2\beta}\int e^{i\kappa r\sin\theta\sin\theta'\cos\alpha'}\frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha'-\cos\frac{\pi}{\beta}(\phi-\phi')}\,d\alpha', \qquad (17)$$

over the *upper* part of the path (A') of Fig. 2 in the α'-plane, or an equivalent path.

Also it is not difficult to show that this function $u(\phi')$ satisfies the equation

$$u(\phi') = \frac{\pi}{\beta}e^{i\kappa r\cos\theta\cos\theta'}$$

$$\times\left\{J_0(\kappa r\sin\theta\sin\theta')+2\sum_1^\infty e^{is\pi^2/2\beta}J_{s\pi/\beta}(\kappa r\sin\theta\sin\theta')\cos s\pi(\phi-\phi')/\beta\right\}. \qquad (18)$$

11. For the physical problem, the solution is given as in §§ 2, 7, and 9, by

$$u = u(\phi')\pm u(-\phi'),$$

according as the boundary condition is $\partial u/\partial n = 0$ or $u = 0$.

Thus we have

$$u = \frac{i}{2\beta}e^{i\kappa r\cos\theta\cos\theta'}\int e^{i\kappa r\sin\theta\sin\theta'\cos\alpha'}$$

$$\times\left\{\frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha'-\cos\frac{\pi}{\beta}(\phi-\phi')}\pm\frac{\sin\frac{\pi}{\beta}\alpha'}{\cos\frac{\pi}{\beta}\alpha'-\cos\frac{\pi}{\beta}(\phi+\phi')}\right\}d\alpha', \qquad (19)$$

over the *upper* part of the path (A′) of Fig. 2 in the α'-plane, or an equivalent path.

And in series, using cylindrical coordinates (ϖ, ϕ, z), we have

$$\left.\begin{aligned} u &= \frac{2\pi}{\beta} e^{i\kappa z \cos\theta'} \\ &\times \left\{ J_0(\kappa\varpi \sin\theta') + 2\sum_1^\infty e^{is\pi^2/2\beta} J_{s\pi/\beta}(\kappa\varpi \sin\theta') \cos s\pi\phi/\beta \cos s\pi\phi'/\beta \right\} \end{aligned}\right\} . \quad (20)$$

or

$$u = \frac{4\pi}{\beta} e^{i\kappa z \cos\theta'} \sum_1^\infty e^{is\pi^2/2\beta} J_{s\pi/\beta}(\kappa\varpi \sin\theta') \sin s\pi\phi/\beta \sin s\pi\phi'/\beta$$

The results in §§ 10 and 11 can be deduced from the solution for the point-source at (r', θ', ϕ'), by taking as fundamental solution*

$$\frac{r'}{R} e^{-i\kappa R},$$

and then letting $r' \to \infty$.

* Cf. Bromwich, *loc. cit.*, p. 461.

ON NON-HARMONIC FOURIER SERIES

By W. H. Young.

[Received March 13th, 1918. — Read March 14th, 1918.]

1. Following English usage, I have found it convenient to restrict the term Fourier series to those trigonometric series which proceed by cosines and sines of integral values of the variable, or variables, and have the further property that their coefficients are expressible by means of integrals of the well-known kind, involving a function of the independent variable, or variables. I have, for example, not followed the usage adopted by Jordan, and other French writers, of employing the term to characterise analogous series of functions, other than $\cos nx$ and $\sin nx$, such as Bessel functions. It seems to me, however, desirable to retain the term when the very slight change—formally speaking—of substituting $n+k$ for n is made. Though the series so obtained have not, however, as far as I know, been the subject of systematic study, and I have found it necessary to obtain among other things, the expressions for the coefficients as integrals, such series naturally present themselves. Special examples of such series are indeed known.* I myself have been led to the study of these series naturally by the necessity of extending to the case when the order is irrational, properties of series of Bessel functions, which, in the rational case, I have been able to deduce from the theory of what I have called Restricted Fourier Series. Needless to say Series of Bessel Functions are not the only series whose study might be expected to demonstrate the same need.

In the researches in question, these non-harmonic trigonometrical series do not present themselves in what I propose to call the Fourier form, being of a more general type. It is evident, however, that a proper grasp of the whole class of trigonometrical series in question cannot be obtained without investigating the properties of the particular class in question. Indeed the behaviour of a Bessel series† at the further bound

* See below, footnote to § 11.

† I distinguish between a series of Bessel functions $\Sigma A_r J_m(k_r z)$ and the special case, Bessel series, when the coefficients A have the appropriate form.

of the interval in which the expansion is valid, is most conveniently discussed by reference to the behaviour of a non-harmonic Fourier series, restricted in character generally, at an extremity of the interval in which this series is defined.

The paper falls naturally into three parts. The first of these, §§ 2–16, treats of the definition and summation of the series in question.

A non-harmonic trigonometrical $\left.\begin{matrix}\text{cosine}\\ \text{sine}\end{matrix}\right\}$ series

$$\sum_{n=-\infty}^{\infty} a_n \begin{matrix}\cos\\ \sin\end{matrix} (n+k)\,x$$

is said to have *the Fourier form* if

$$a_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \begin{matrix}\sin\\ \cos\end{matrix} (n+k)(2\pi-t)\,dt,$$

and the function $f(x)$ is then said to be *the associated function* of the series.

At an internal point of the interval $(0, 2\pi)$ these series are proved to have, apart from an additive term which tends uniformly to zero, the same expression for the n-th partial summation

$$s_n = o(1) + \frac{1}{\pi} \int_0^c \tfrac{1}{2} \{f(x+t)+f(x-b)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})t\,dt,$$

as the Fourier series of a function equal to $f(x)$ at, and in the neighbourhood of, the point x, and having any convenient values elsewhere. *The upper and lower functions, and the modes of oscillation, are accordingly independent of the form of the associated function, except in an arbitrary small neighbourhood of the point x considered, and are the same as those of this auxiliary Fourier series. In particular, the conditions of convergence, and of uniform or bounded convergence, are the same for the non-harmonic Fourier series and the auxiliary harmonic Fourier series.*

These results are still true at the end-points of the interval $(0, 2\pi)$, provided the function $f(x)$, whose values have hitherto only been supposed known almost everywhere in the interval $(0, 2\pi)$, is supposed " continued " outside the interval in accordance with certain laws, which are different for the cosine and sine series. The law of " continuation " appropriate to the cosine is embodied in the formulæ :

$$\left.\begin{aligned} f(t) &= f(-t),\\ \tfrac{1}{2}\{f(2\pi+t)+f(2\pi-t)\} &= f(t)\cos 2k\pi. \end{aligned}\right\}$$

The law of "continuation" appropriate to the sine series is as follows:

$$\left.\begin{aligned} f(t) &= -f(-t), \\ \tfrac{1}{2}\{f(2\pi+t)-f(2\pi-t)\} &= f(t)\cos 2k\pi. \end{aligned}\right\}$$

These "continuation" formulæ play in our present theory the part of the periodicity in the case of harmonic series.

The proof of these properties depends on certain fundamental lemmas, which, in our theory, take the place of the ordinary normal properties of the sine and cosine. These are

$$\int_0^{2\pi} \begin{matrix}\sin\\ \cos\end{matrix} (n+k)(2\pi-u) \begin{matrix}\cos\\ \sin\end{matrix} (r+k)u\,du = 0 \quad (r \neq n)$$

or

$$= \pm\pi \sin 2k\pi \quad (r = n),$$

r and n being integers, positive, negative, or zero. In particular it appears that, if a function is expressible as the sum of either a convergent non-harmonic Fourier cosine series, or the corresponding sine series, at a point x internal to the interval $(0, 2\pi)$, it is expressible by both these series; but that this is not the case at the points 0 and 2π, unless the value of the function at each of these points is zero. At the origin indeed a sine series, of course, converges to zero, and at 2π each term of a cosine series returns to its value at the origin, multiplied by $\cos 2k\pi$.

It follows from the theorem of Riemann-Lebesgue, that the coefficients of a non-harmonic Fourier series tend to zero as we advance along the series. Hence the second integrated series necessarily converges uniformly to a continuous function. We are thus able to show at once that *the necessary and sufficient condition that a non-harmonic* $\begin{matrix}\textit{cosine}\\ \textit{sine}\end{matrix}$ *series should have the Fourier form is that the second integrated series should converge to a second integral,* and the associated function is then the second differential coefficient of this second integral almost everywhere in* $(0, 2\pi)$: *also, if we prefer to utilise at once both types of series, that we may in this statement replace "second" by "first."*

The first part of the paper terminates with some simple examples of non-harmonic Fourier cosine and sine series, the nature of whose convergence, uniform or bounded, is discussed.

The object of the second part of the paper, §§ 17–25, is mainly to shew that there is no other way of obtaining the development of a given function in such a series as is here contemplated. For this and later

* That is the integral of an integral.

purposes we require the generalisation of Riemann's "Three Theorems on Trigonometrical Series," given in his *Habilitationsschrift.*

Harnack's theorem, too, as to certain conditions under which the coefficients of a trigonometrical series tend to zero is generalised so as to apply to non-harmonic series. In particular we have the result that, *if the points of non-convergence form a set which is at most of the first category in some interval, then the coefficients tend to zero.*

One of the most important consequences of these theorems is the result we were in search of: *no two distinct non-harmonic* $\left.\begin{matrix}\text{cosine}\\ \text{sine}\end{matrix}\right\}$ *series of the type* $\sum_{n=-\infty}^{\infty} a_n \begin{matrix}\cos\\ \sin\end{matrix} (n+k)x$, *with the same* k, *which converge to the same value at each point of* $(0, 2\pi)$ *with the exception at most of a countable set of points* can exist.*

The main object of the third part of the paper (§§ 26 to end), is to apply the considerations which precede to establish rigidly a statement which I made in a previous paper† as to the behaviour of a Bessel series at the point $z = 1$. For this purpose it is convenient to use the theory of restricted non-harmonic Fourier series in a slightly extended form. In the light of the continuation formulæ this theory is found to apply not only when the interval of restriction belongs to the completely open interval $(0, 2\pi)$, but even when one of these points is an included end-point of the interval of restriction.

In consequence we may now state that *the conditions of convergence of a Bessel series at any point of the half-open interval* $(0 < x \leqslant 1)$ *are the same as those of the Fourier series of a function equal to the associated function* $f(z)$ *of the Bessel series in an arbitrary small neighbourhood surrounding the point considered and having any convenient values elsewhere, provided the value assigned to* $f(z)$ *at any point to the right of the point* $z = 2\pi$ *is defined as the same as that at the reflection of this point in* $z = 2\pi$.

2. I begin by stating the following lemmas, which take the place of the familiar normal properties of the sine and cosine in that of Fourier series :—

LEMMA 1.—*If* r *and* n *are different integers, positive, negative, or zero,*

$$\int_0^{2\pi} \sin(n+k)(2\pi-u)\cos(r+k)u\,du = 0 \quad (0 < k < 1),$$

* Or totally imperfect set.

† "On Series of Bessel Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 163–200.

and if $r = n$, *the value of the integral is*

$$\pi \sin 2k\pi.$$

LEMMA 2.—*If* r *and* n *are different integers, positive, negative, or zero,*

$$\int_0^{2\pi} \cos(n+k)(2\pi-u) \sin(r+k)\, u\, du = 0,$$

and, if $r = n$, *the value of the integral is*

$$-\pi \sin 2k\pi.$$

The proofs may be left to the reader.

3. Hence, in place of the ordinary Fourier expansion, we are led to examine the possibility of expanding a function $f(x)$ in a series of the form*

$$\sum_{n=-\infty}^{\infty} a_n \cos(k+n)\, x, \tag{1}$$

where, in view of Lemma 1, the coefficients a_n have what may be called the Non-harmonic Fourier Form,

$$a_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \sin(n+k)(2\pi-t)\, dt, \tag{2}$$

for all integral values of n, positive, negative, or zero, and $f(t)$ is supposed defined and summable only in the closed interval $(0, 2\pi)$, to which we restrict our attention. When the coefficients have this form, in which case $a_n \to 0$, by the theorem of Riemann-Lebesgue, we shall write symbolically

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(k+n)\, x \quad (0 \leqslant x \leqslant 2\pi). \tag{3}$$

We shall then find that the partial summation s_n, that is

$$s_n(x) = \sum_{r=-n}^{n} a_r \cos(k+r)x,$$

apart from an additive term which tends uniformly to zero, has the same form as that of the Fourier series of a function $\phi_x(t)$, equal to $f(t)$ in an arbitrary small interval enclosing the point x considered, and having any convenient values elsewhere, provided x is internal to the completely open interval $(0, 2\pi)$.

---

* This symbol is to be understood to mean $\operatorname{Lt}_{n\to\infty} \sum_{r=-n}^{n} a_r \cos(k+r)\, x$.

At the point 0 the same is still true, provided $f(x)$ be regarded as "continued" beyond the origin on the left, so as to be an even function in the arbitrary small interval in question.

At the point 2π, the formula for s_n only differs from that at 0 by the multiplicative factor $\cos 2k\pi$, and the connection with the Fourier series of $\phi_x(t)$ still holds true, provided $f(t)$ be regarded as "continued" beyond the point 2π on the right, in accordance with the relation

$$\tfrac{1}{2}\{f(2\pi+t)+f(2\pi-t)\} = f(t)\cos 2k\pi, \tag{4}$$

in the arbitrary small interval in question. It will be noticed that the "continuation" defined in the present article is precisely that which is fulfilled of itself when the non-harmonic series converges to $f(x)$ everywhere.

4. To prove these formulæ, we have

$$s_n = \frac{1}{\pi \sin 2k\pi} \sum_{r=-n}^{n} \int_0^{2\pi} \{\cos(k+r)\,x \sin(k+r)(2\pi-t)\}\, f(t)\,dt$$

$$= \frac{1}{2\pi \sin 2k\pi} \sum_{r=-n}^{n} \int_0^{2\pi} \{\sin(k+r)(x+t) - \sin(k+r)(x-t)\}\, f(2\pi-t)\,dt \quad \text{(A)}$$

$$= \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t) \left[\sin k(x+t) \left\{\tfrac{1}{2} + \sum_{r=1}^{n} \cos r(x+t)\right\}\right.$$

$$\left. - \sin k(x-t) \left\{\tfrac{1}{2} + \sum_{r=1}^{n} \cos r(x-t)\right\}\right] dt$$

$$= \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t) \left[\sin k(x+t) \operatorname{cosec} \tfrac{1}{2}(x+t) \sin (n+\tfrac{1}{2})(x+t)\right.$$

$$\left. - \sin k(x-t) \operatorname{cosec} \tfrac{1}{2}(x-t) \sin (n+\tfrac{1}{2})(x-t)\right] dt. \quad \text{(A')}$$

5. Now suppose first that

$$0 < x < 2\pi,$$

then the integrand is the product of $f(2\pi-t)$ into the sum of two terms, each of the form

$$u(x, t) \sin nt + v(x, t) \cos nt,$$

where, in the first of the two terms, u and v are bounded, except in the neighbourhood of $t = 2\pi - x$, and, in the second, without exception, since $\sin k(x-t) \operatorname{cosec} \frac{1}{2}(x-t)$ remains bounded in the neighbourhood of the point $t = x$.

Thus, by the theorem of Riemann-Lebesgue, we may write

$$s_n = o(1) + \frac{1}{2\pi \sin 2k\pi} \int_{2\pi-x-e}^{2\pi-x+e} f(2\pi-t) \sin k(x+t) \operatorname{cosec} \tfrac{1}{2}(x+t) \sin(n+\tfrac{1}{2})(x+t)\, dt$$

$$= o(1) + \frac{1}{2\pi \sin 2k\pi} \int_0^e \{f(x+t) \sin k(2\pi-t) + f(x-t) \sin k(2\pi+t)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= o(1) + \frac{1}{2\pi} \int_0^e \{f(x+t)+f(x-t)\} \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt\,;$$

for, by the theorem of Riemann-Lebesgue, the integral which we have neglected tends to zero, since $\cos 2k\pi \sin kt \operatorname{cosec} \frac{1}{2}t$ remains bounded throughout $(0, e)$.

Now in the last integral we may replace $\cos kt$ by unity, since this integral may be broken up into two parts, differing from the whole only in having unity and $-1+\cos kt$ respectively in place of $\cos kt$, and the latter part tends to zero by the theorem of Riemann-Lebesgue.

Thus, finally,

$$s_n(x) = o(1) + \frac{1}{2\pi} \int_0^e \{f(x+t)+f(x-t)\} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt, \qquad (5)$$

which is the well-known Fourier formula.

This proves the first of our statements (§ 3).

6. Returning to (A′) and putting $x = 0$, we get

$$s_n(0) = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t) \sin kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= o(1) + \frac{1}{\pi \sin 2k\pi} \int_{2\pi-e}^{2\pi} f(2\pi-t) \sin kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2}) t\, dt,$$

by the theorem of Riemann-Lebesgue,

$$= o(1) + \frac{1}{\pi \sin 2k\pi} \int_0^e f(t) \sin k(2\pi-t) \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt.$$

Expanding $\sin k(2\pi-t)$ and using again the theorem of Riemann-Lebesgue, we get

$$= o(1) + \frac{1}{\pi} \int_0^e f(t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt$$

$$= o(1) + \frac{1}{\pi} \int_0^e f(t) \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, dt, \qquad (5')$$

as at the end of the preceding article.

This proves the statement made (§ 3) as to the identity of this formula with that corresponding to the Fourier series of an even function $\phi_0(t)$ equal to $f(t)$ in $(0, e)$.

Thus, *if $f(x)$ is continuous on the right at the origin, the series converges there to $f(0)$.*

7. The statement as to the formula at the point $x = 2\pi$, follows at once. Putting $x = 2\pi$ in formula (A′) it becomes

$$\begin{aligned} s_n(2\pi) &= \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi - t) \cos 2k\pi \sin kt \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\, t \\ &= \cos 2k\pi\, s_n(0) \\ &= o(1) + \frac{1}{\pi} \int_0^e f(t) \cos 2k\pi \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\, t\, dt \\ &= o(1) + \frac{1}{2\pi} \int_0^e \{f(2\pi - t) + f(2\pi + t)\} \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\, t\, dt, \quad (5'') \end{aligned}$$

supposing $f(t)$ "continued" in accordance with the law formulated in equation (4).

The series therefore converges at 2π if, and only if, it converges at the origin, and the value is $\cos 2k\pi$ *times the value at the origin.*

8. By the formulæ (5), (5′) and (5″) we have at once the following theorem :—

THEOREM.—*The non-harmonic Fourier cosine series*

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos (n+k)x, \qquad (3)$$

where

$$a_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \sin (n+k)(2\pi - t)\, dt, \qquad (2)$$

converges at any point x of the closed interval $(0, 2\pi)$ under the same conditions, and to the same value, as the Fourier series of a function $\phi_x(t)$, equal to $f(t)$ in an arbitrary small neighbourhood enclosing the point x in question and having any convenient values elsewhere, provided $f(t)$ be supposed "continued" outside the interval in question, so that, in an arbitrary small neighbourhood of the origin it is an even function, and in an arbitrary small neighbourhood of the point 2π,

$$\tfrac{1}{2}\{f(2\pi - t) + f(2\pi + t)\} = f(t) \cos 2k\pi. \qquad (4)$$

We remark that it follows that the non-harmonic Fourier series cannot represent the function $f(t)$ at the point 2π unless

$$f(2\pi) = f(+0)\cos 2k\pi, \tag{6}$$

nor at the origin unless $f(t)$ is continuous on the right.

9. Remembering that the theorem of Riemann-Lebesgue asserts, not merely that a certain integral tends to zero, but that it does so uniformly, we may combine the formulæ (5), (5′), and (5″) in the formula

$$s_n(x) - \sigma_n(x) = \eta_n(x), \tag{7}$$

where $\sigma_n(t)$ denotes the n-th partial summation of the Fourier series of the auxiliary function $\phi_x(t)$, and $\eta_n(x)$ tends uniformly to zero.

Now by the definition of the auxiliary function $\phi_x(t)$ as equal to $f(t)$ in an arbitrary small neighbourhood enclosing the point x and having any convenient values elsewhere, we see that, taking $|h| < e$, we have for all these values of h, choosing the auxiliary functions conveniently,

$$\phi_{x+h}(t) \equiv \phi_x(t),$$

t lying in the arbitrary small neighbourhood $(x-e, x+e)$ already utilised in constructing $\phi_x(t)$. Hence we may write

$$s_n(x+h) - \sigma_n(x+h) = \eta_n(x+h),$$

where $\eta_n(x+h)$ tends uniformly to zero. Therefore

$$\{s_m(x) - s_n(x+h)\} - \{\sigma_m(x) - \sigma_n(x+h)\} = \eta_m(x) - \eta_n(x+h). \tag{8}$$

This shews that if one of the series converges uniformly, or boundedly, at the point x, on the right, or on the left, so does the other.

If the series do not converge, we see, by (7) and (8), that they have the same upper and lower functions, and that, if one of them oscillates boundedly on the right or on the left, so does the other, while if one of them oscillates uniformly (in the first mode) above, or below, on the right, or on the left, so does the other.

In particular if $f(2\pi) = \cos 2k\pi \,.\, f(0)$,

and $f(x)$ is an integral in the closed interval $(0, 2\pi)$, the non-harmonic Fourier cosine series of (x) converges uniformly to $f(x)$ in the closed interval $(0, 2\pi)$.

10. This last result enables us to prove that *the necessary and sufficient condition that a non-harmonic cosine series should have the Fourier form*

is that the second integrated series should converge to a second integral, that is to the integral of an integral, in the closed interval $(0, 2\pi)$, *and the associated function of the original series is then almost everywhere equal to the second differential coefficient of that repeated integral.**

Indeed writing $$\int_0^x f(x)\,dx = F(x),$$

$$a + \int_0^x F(x)\,dx = G(x),$$

we shall have $$G(2\pi) = \cos 2k\pi \,.\, G(0),$$

if a has the value $$a = -\frac{1}{1-\cos 2k\pi}\int_0^{2\pi} F(x)\,dx.$$

Now let $$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(n+k)x,$$

$$G(x) \sim \sum_{n=-\infty}^{\infty} A_n \cos(n+k)x\,;$$

then the latter series converges uniformly to $G(x)$ in the closed interval $(0, 2\pi)$.

But

$$\begin{aligned}
&-\pi \sin 2k\pi (k+n)^{-2} a_n \\
&\quad = -\int_0^{2\pi} (k+n)^{-2} f(x) \sin (k+n)(2\pi - x)\,dx \\
&\quad = -\int_0^{2\pi} (k+n)^{-2} \sin(k+n)(2\pi - x)\,dF(x) \\
&\quad = -\Big[(k+n)^{-2} \sin(k+n)(2\pi - x)\,F(x)\Big]_0^{2\pi} \\
&\qquad\qquad -\int_0^{2\pi} (k+n)^{-1} F(x) \cos(k+n)(2\pi - x)\,dx \\
&\quad = -\int_0^{2\pi} (k+n)^{-1} \cos(k+n)(2\pi - x)\,dG(x) \\
&\quad = -\Big[(k+n)^{-1} G(x) \cos(k+n)(2\pi - x)\Big]_0^{2\pi} + \int_0^{2\pi} G(x) \sin(k+n)(2\pi - x)\,dx \\
&\quad = -(k-n)^{-1}\{G(2\pi) - \cos 2k\pi G(0)\} + A_n \\
&\quad = A_n.
\end{aligned}$$

* It will be seen below that in this statement we may replace the second by the first.

The relation between the series

$$\sum_{n=-\infty}^{\infty} a_n \cos(n+k)x \quad \text{and} \quad \sum_{n=-\infty}^{\infty} A_n \cos(n+k)x$$

is therefore that the second is the second integrated series of the first.

Thus the second integrated series of the non-harmonic Fourier cosine series of $f(x)$ converges uniformly to a second integral of $f(x)$, and conversely, the second differentiated series of the non-harmonic Fourier cosine series of a second integral of $f(x)$ is the non-harmonic Fourier cosine series of $f(x)$.

The result stated at the beginning of this article is equivalent to this last double statement and is therefore proved.

11. *Example.*—Put $f(t) = 1$.

We get for all integral values of n, positive, negative, and zero,

$$a_n \pi \sin 2k\pi = (1-\cos 2k\pi)/(k+n),$$

that is,

$$a_n \pi \cot k\pi = 1/(k+n).$$

The "continuation" of the function gives us

$$f(-t) = 1, \quad f(2\pi+t) = 2\cos 2k\pi - 1,$$

where $0 < t < e$.

Thus at each point the auxiliary Fourier series converges, and it converges uniformly except at $x = 2\pi$, where it still converges boundedly. The sum is 1 except at $x = 2\pi$, where it is $\cos 2k\pi$.

Hence, by our general theory the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{k+n} \cos(k+n)x$$

converges uniformly to $\pi \cot k\pi$ at each point of $(0, 2\pi)$, except at the point 2π, where it converges boundedly to $\pi \cot k\pi \cos 2k\pi$.

Thus* in $0 \leqslant x < 2\pi$ we have, uniformly,

$$\pi \cot k\pi = \sum_{n=-\infty}^{\infty} \frac{1}{k+n} \cos(k+n)x$$

$$= \frac{\cos kx}{k} + 2\sum_{n=1}^{\infty} \frac{k\cos kx \cos nx + n \sin kx \sin nx}{k^2-n^2}. \qquad (9)$$

* This result has already been obtained (for $0 < x < 2\pi$) by a direct method. See Bromwich, *Theory of Infinite Series*, § 90, p. 231, top, where references are supplied.

Putting $x = 0$ in this equation, we get the well-known formula

$$\pi \cot k\pi = \sum_{n=-\infty}^{\infty} \frac{1}{k+n}. \tag{10}$$

We get the same formula, with each side multiplied by $\cos 2k\pi$, when $x = 2\pi$.

When $x = \pi$, we get the known cosecant formula.

That the convergence is bounded is evident from the series itself, as well as that it is uniform, except at $x = 2\pi$, for

$$2k \cos kx \sum_{n=1}^{\infty} \frac{\cos nx}{k^2-n^2} \quad \text{and} \quad 2k \sin kx \sum_{n=1}^{\infty} \frac{\sin nx}{k^2-n^2}$$

converge uniformly for all values of x, while

$$-2 \sin kx \sum_{n=1}^{\infty} \frac{(k-n) \sin nx}{k^2-n^2} = 2 \sin kx \sum_{n=1}^{\infty} \frac{\sin nx}{k+n}.$$

Now, by Abel's lemma, the latter series converges uniformly for all values of x in $(0, 2\pi)$ excluding the end-points 0 and 2π. Also the series is known to converge boundedly in $(0 \leqslant x \leqslant 2\pi)$, and therefore, when multiplied by $\sin kx$, converges at the origin uniformly to zero. Thus our series, being the sum of these three series, converges boundedly, and, except at $x = 2\pi$, uniformly.

12. Again, in virtue of Lemma 2, we are led to examine the possibility of expanding a function $f(x)$ in a series of the form

$$\sum_{n=-\infty}^{\infty} b_n \sin(k+n)x, \tag{11}$$

where, in view of Lemma 2, the coefficients have what may be called the non-harmonic Fourier form,

$$b_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} f(t) \cos (n+k)(2\pi-t)\, dt, \tag{12}$$

for all integral values of n, positive, negative, and zero : $f(t)$ being supposed defined and summable only in the closed interval $(0, 2\pi)$, to which we restrict our attention.

We again write symbolically

$$f(x) \sim \sum_{n=-\infty}^{\infty} b_n \sin(k+n)x, \tag{13}$$

where, by the theorem of Riemann-Lebesgue, $b_n \to 0$ $(n \to \infty)$.

We then find the same result as before, in so far as the internal points of the interval $(0, 2\pi)$ are concerned. We have, however, in order to extend the result to the end-points 0 and 2π to use a different mode of "continuation" from that applicable to the non-harmonic Fourier cosine series. This is again that which is naturally fulfilled when the series converges everywhere to $f(x)$. Indeed the partial summation s_n of the sine series (11) has the same form (A) as before, except that the two terms inside the bracket are now united by the sign + instead of −,

$$s_n = \frac{1}{2\pi \sin 2k\pi} \sum_{r=-n}^{n} \int_0^{2\pi} \{\sin(k+r)(x+t) + \sin(k+r)(x-t)\} f(2\pi-t)\,dt. \quad \text{(B)}$$

This only affects the term which, at the next stage, disappears by the theorem of Riemann-Lebesgue. We get, in fact, corresponding to the formula (A′),

$$s_n' = \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t)\,[\sin k(x+t)\operatorname{cosec} \tfrac{1}{2}(x+t) \sin (n+\tfrac{1}{2})(x+t) + \sin k(x-t)\operatorname{cosec} \tfrac{1}{2}(x-t) \sin (n+\tfrac{1}{2})(x-t)]\,dt, \quad \text{(B′)}$$

which leads as before to the formula (5).

13. At the origin $s_n = 0$, always, which implies the law of "continuation" by which $f(t)$ is an odd function in an arbitrary small neighbourhood enclosing the origin.

14. At $x = 2\pi$, we have, using (A′), after changing the − into +,

$$s_n(2\pi) = \frac{1}{2\pi \sin 2k\pi} \int_0^{2\pi} f(2\pi-t)\{\sin k(2\pi+t) + \sin k(2\pi-t)\} \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\,t\,dt$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(2\pi-t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\,t\,dt$$

$$= o(1) + \frac{1}{\pi} \int_0^{e} f(2\pi-t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\,t\,dt + \frac{1}{\pi} \int_{2\pi-e}^{2\pi} f(2\pi-t) \cos kt \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\,t\,dt$$

$$= o(1) + \frac{1}{\pi} \int_0^{e} \{f(2\pi-t) + f(t) \cos k(2\pi-t)\} \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\,t\,dt$$

$$= o(1) + \frac{1}{\pi} \int_0^{e} \{f(2\pi-t) + f(t) \cos 2k\pi\} \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\,t\,dt$$

using the theorem of Riemann-Lebesgue in the third and fifth of these equations.

Thus if we "continue" our function $f(t)$ beyond the right-hand end-point 2π, according to the law

$$\tfrac{1}{2}\{f(2\pi+t)-f(2\pi-t)\} = f(t)\cos 2k\pi, \tag{14}$$

we shall have

$$s_n(2\pi) = o(1)+\frac{1}{2\pi}\int_0^e \{f(2\pi-t)+f(2\pi+t)\}\operatorname{cosec}\tfrac{1}{2}t\sin(n+\tfrac{1}{2})t\,dt, \tag{5''}$$

which is the Fourier formula.

15. Hence we have the following theorem:—

THEOREM.—*The non-harmonic Fourier sine series*

$$f(x) \sim \sum_{n=-\infty}^{\infty} b_n \sin(n+k)x, \tag{13}$$

where

$$b_n = \frac{1}{\pi \sin 2k\pi}\int_0^{2\pi} f(t)\cos(n+k)(2\pi-t)\,dt, \tag{12}$$

converges at any point x of the closed interval $(0, 2\pi)$ under the same conditions, and to the same value, as the Fourier series of a function $\phi_x(t)$, equal to $f(t)$ in an arbitrary small neighbourhood enclosing the point x in question and having any convenient values elsewhere, provided $f(t)$ be supposed "continued" outside the interval in question, so that, in an arbitrary small neighbourhood of the origin it is an odd function, and in an arbitrary small neighbourhood of the point 2π,

$$\tfrac{1}{2}\{f(2\pi+t)-f(2\pi-t)\} = f(t)\cos 2k\pi. \tag{14}$$

We remark that it follows that the non-harmonic Fourier series *cannot represent the function $f(t)$ at the origin unless*

$$f(0) = 0, \tag{15}$$

and it cannot represent the function $f(t)$ at the point 2π, unless

$$f(2\pi) = f(2\pi-0)+f(+0)\cos 2k\pi. \tag{16}$$

If therefore continuous on the right at the origin, it must be continuous on the left at 2π.

The discussion of the uniform, or bounded, convergence or oscillation, on the right, or on the left, given in § 9, evidently applies equally to the sine series, the appropriate law of "continuation" being supposed utilised.

In particular, if $f(0) = 0$, *and* $f(x)$ *is an integral in the closed interval* $(0, 2\pi)$, *in which case* (16) *holds, then the non-harmonic Fourier sine series of* $f(x)$ *converges uniformly to* $f(x)$ *in the closed interval* $(0, 2\pi)$.

Thus, as an application of this result, we see that if

$$F(x) = \int_0^x f(x)\,dx \sim \sum_{n=-\infty}^{\infty} b_n \sin(k+n)x,$$

where

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n \cos(k+n)\,x,$$

the non-harmonic Fourier sine series of $F(x)$ converges uniformly in $(0, 2\pi)$ to $F(x)$. But

$$\pi \sin 2k\pi (k+n)^{-1} a_n = \int_0^{2\pi} (k+n)^{-1} \sin(k+n)(2\pi - x)\, f(x)\,dx$$

$$= \int_0^{2\pi} (k+n)^{-1} \sin(k+n)(2\pi - x)\,dF(x)$$

$$= \int_0^{2\pi} F(x) \cos(k+n)(2\pi - x)\,dx = b_n.$$

Thus we see that *the first integrated series of a non-harmonic Fourier cosine series converges uniformly in the closed interval* $(0, 2\pi)$ *to the integral of the associated function of the cosine series; and, conversely, the first derived series of the non-harmonic Fourier sine series of an integral is a non-harmonic Fourier cosine series, whose associated function has that integral for integral.*

Similarly, using the result proved in § 9, that the non-harmonic cosine series of an integral which satisfies the proper "continuation" law, converges uniformly to that integral in the closed interval $(0, 2\pi)$, we see that in the preceding statement in *italics* we may write *cosine* for *sine*.

We may state these results otherwise by saying: *The necessary and sufficient condition that a non-harmonic sine or cosine series should have the Fourier form is that the first integrated series should converge towards an integral in the closed interval* $(0, 2\pi)$.

We have seen that a similar condition may be stated in the case of the cosine series, and may evidently also be given for the sine series, in which the second integrated series takes the place of the first in this statement, the integral (first integral) being replaced by a second integral. This form of the condition has the advantage of only comparing sine series with sine series, and cosine series with cosine series.

16. As an example, suppose that

$$f(x) = 1.$$

We get for all integral values of n, positive, negative, and zero,

$$b_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \cos(k+n)(2\pi - t)\, dt = \frac{1}{\pi(k+n)}.$$

The "continuation" according to the law appropriate to the sine series, gives us $f(-t) = 1$, and $f(2\pi+t) = 2\cos 2k\pi + 1$. Thus at the origin and at the point 2π the auxiliary Fourier series, which is everywhere convergent, does not represent the function. The convergence is uniform except at the end-points, where it is bounded. Hence, for $0 < x < 2\pi$, we have*

$$\pi = \sum_{n=-\infty}^{\infty} \frac{\sin(k+n)x}{k+n} = \frac{\sin kx}{k} + \sum_{n=1}^{\infty} \frac{k \sin kn \cos nx - n \cos kn \sin nx}{k^2 - n^2}, \quad (17)$$

the convergence being uniform, while at the origin the series converges boundedly to zero, and, at the point 2π, it converges boundedly to $\pi\{1+\cos 2k\pi\} = 2\pi \cos^2 k\pi$, agreeing with (10).

17. We next proceed to generalise Riemann's three theorems on trigonometrical series.† Riemann's reasoning applies with comparatively slight alterations to non-harmonic trigonometrical series, and in order to save space I have not written out the proofs at length.

RIEMANN'S FIRST THEOREM (*generalised*).—

If the series $\sum_{n=-\infty}^{\infty} A_n^{(k)}$, *where*

$$A_n^{(k)} = a_n \cos(n+k)x + b_n \sin(n+k)x,$$

converges at the point x to, say, $f(x)$, and its second integrated series converges at x and in its neighbourhood to a function $G(x)$, then

$$\{G(x+\alpha) - 2G(x) + G(x-\alpha)\}/\alpha^2 \to f(x) \quad (\alpha \to 0).$$

---

* See footnote to § 11.

† B. Riemann, *Habilitationsschrift*, § 8, *Ges. Werke*, pp. 246–249.

We have

$$\{G(x+2\alpha)-2G(x)+G(x-2\alpha)\}/4\alpha^2$$

$$=-\sum_{n=-\infty}^{\infty}(k+n)^{-2}a_n\{\cos(k+n)(x+2\alpha)-2(k+n)\cos x+\cos(k+n)(x-2\alpha)\}$$

$$=\sum_{n=-\infty}^{\infty}A_n\left\{\frac{\sin(k+n)\alpha}{(k+n)\alpha}\right\}^2. \tag{18}$$

From this point on the proof is, in all essentials, identical with Riemann's.*

COR.—*Under the same circumstances if α and β approach zero so that the ratio of each to the other does not tend towards zero*

$$\frac{G(x+\alpha+\beta)-G(x+\alpha-\beta)-G(x-\alpha+\beta)+G(x-\alpha-\beta)}{4\alpha\beta}\to f(x).$$

This is the form in which Riemann enunciates his theorem, but we use it in the simpler form, from which as he points out, it immediately follows.

18. RIEMANN'S SECOND THEOREM (*generalised*).—

If the second integrated series of the series

$$\sum_{n=-\infty}^{\infty}A_n\equiv\sum_{n=-\infty}^{\infty}\{a_n\cos(n+k)x+b_n\sin(n+k)x\},$$

converges at x and in its neighbourhood to a function $G(x)$, then

$$\{G(x+\alpha)-2G(x)+G(x-\alpha)\}/\alpha\to 0\quad(\alpha\to 0).$$

The proof of this theorem also is practically identical with Riemann's.

---

* Similarly we may shew that *when the series* $\sum_{n=-\infty}^{\infty}A_n$ *does not converge, all the limits of*

$$\{G(x+h)-2G(x)+G(x-\alpha)\}/\alpha^2,$$

when $\alpha\to 0$, *lie between the two bounds*

$$\tfrac{1}{2}(U+L)\pm\tfrac{1}{2}(U-L)\left(1+\frac{2}{\pi^2}\right),$$

where U and L are the values of the upper and lower functions of $\sum_{-\infty}^{\infty}A_n$ *at the point x, provided only the second integrated series converge and $G(x)$ denote its sum.*

This form of the result is due to my wife, being a closer pair of bounds than that given in the case of harmonic series by Hassenfelder, "Zur Theorie der trigonometrischen Reihe" (1900), *Jahresbericht des königlichen Gymnasiums zu Strassburg* (Teubner), as an emendation of a similar result due to Du Bois Reymond. See a forthcoming paper by Grace Chisholm Young, also Hobson's *Theory of Functions of a Real Variable.*

19. The extended form of Riemann's Third Theorem, I have already given in the recent communication to the Royal Society already cited.* The enunciation is as follows:—

RIEMANN'S THIRD THEOREM (*generalised*).—

If in (b, c) *the function* $\lambda(x)$ *is the integral of an integral, and if* $\lambda(x)$ *and* $\lambda'(x)$ *are both zero at* b *and* c, *then*

$$I \equiv \mu^2 \int_b^c \{G(x+u)+G(x-u)\}\,\lambda(u) \begin{matrix}\cos\\ \sin\end{matrix} \mu u\,du \to 0 \quad (\mu \to \infty),$$

where

$$G = -\sum_{n=-\infty}^{\infty} (n+k)^{-2}\{a_n \cos(n+k)x + b_n \sin(n+k)x\} = -\sum_{n=-\infty}^{\infty} (n+k)^{-2} A_n^{(k)},$$

where $(0 \leqslant k < 1)$, *provided only* (i) *the series converge uniformly, and* (ii) $A_n^{(k)} \to 0$ $(n \to \infty)$.

The proof, which also is closely modelled on Riemann's will not be introduced here.

20. Harnack has proved the following result:—

If (a, b) *be any interval, and* $r_1, r_2, \ldots$ *any sequence of continually increasing positive integers, we can find a sub-sequence* $n_1, n_2, \ldots$ *and a point* x, *internal to* (a, b), *such that, from and after a certain index, the values*

$$n_i x,$$

all differ from an odd multiple of $\frac{1}{2}\pi$ *by less than an assigned (arbitrary) small positive number* ξ.†

Hence it immediately follows that if at every internal point x of (a, b), we have

$$|\lambda_{n_i} \sin n_i x| < \delta,$$

for all indices i, we must have for all indices i from and after a certain integer,

$$|\lambda_{n_i}| < \delta \sec \xi,$$

and therefore

$$\operatorname*{Llt}_{i \to \infty} |\lambda_{n_i}| \leqslant \delta \sec \xi \leqslant \delta,$$

since ξ is arbitrary, and may be chosen as small as we please.

From this result Harnack easily deduces a theorem for harmonic

* "On Non-harmonic Trigonometrical Series", 1918, § 10.

† The proof may be consulted in Hobson's *Theory of Functions of a Real Variable*, p. 747.

trigonometric series, equally true for non-harmonic series. His proof is equally valid in the more general case. The theorem is as follows:—

Harnack's Theorem (*generalised*).—

If $\sum_{-n}^{n} A_n^{(k)}(x)$ *is a non-harmonic or harmonic trigonometric series, where*

$$A_n^{(k)} = a_n \cos(n+k)\,x + b_n \sin(n+k)\,x \quad (0 \leqslant k < 1),$$

and the series is such that in $(0, 2\pi)$ *there is a sub-interval in which the upper and lower functions at each point* x *differ by less than* δ, *then all the limits of the coefficients* a_n *and* b_n, *when* $n \to \infty$, *are numerically less than or equal to* δ.

Cor. 2.—*If the points of non-convergence form a set which is at most of the first category in some interval, then the coefficients tend towards zero.*

For the points at which the excess of the upper function over the lower function is greater than δ form, as is known, an inner limiting set, and therefore form in every interval in which they are dense everywhere a set of the second category, and cannot therefore be a sub-set of a set of the first category; they must be nowhere dense therefore in an interval in which the points of non-uniform convergence form a set of the first category. Thus the conditions of the corollary hold good, since this is true for all values of δ.

21. To prove the uniqueness of the expansion in a non-harmonic Fourier cosine or sine series, we state and prove the theorem in a form which is practically the most general possible, though, by employing a less simple method the words "countable set of points" might be replaced by "totally imperfect set," that is a set containing no perfect subset. We give the proof for the cosine series, but it will be seen that we may in it substitute the sine for the cosine. The theorem is as follows:—

Theorem.—*No two distinct non-harmonic* $\left\{\begin{matrix} cosine \\ sine \end{matrix}\right.$ *series of the type*

$\sum_{n=-\infty}^{\infty} a_n \cos(n+k)\,x$, *with the same* k, *which converge to the same value at each point of* $(0, 2\pi)$, *with the exception at most of a countable set of points, can exist.*

Suppose, if possible, that there were two such series, and let their

difference be denoted by $\sum_{n=-\infty}^{\infty} a_n \cos(n+k)x$: then this latter series must converge to zero, except at a countable set of points.

Since a countable set is a set of the first category, it follows from the Cor. 2 of § 20, that the coefficient a_n tends to zero.

If we then write

$$G(x) = -\sum_{n=-\infty}^{\infty} (n+k)^{-2} a_n \cos(n+k)x,$$

we see that $G(x)$ is a continuous function, satisfying the two conditions that

$$\text{(i)} \quad \{G(x+t)+G(x-t)-2G(x)\} = 0,$$

except at the countable set of points; and further, without exception

$$\text{(ii)} \quad \{G(x+t)+G(x-t)-2G(x)\}\, t \to 0 \quad (t \to 0).$$

Hence by the extension of Schwartz's theorem,* $G(x)$ is a linear function of x, and we may write

$$Ax+B = -\sum_{n=\infty}^{\infty} (n+k)^{-2} a_n \cos(n+k)x,$$

the convergence being uniform.

Multiply by $\sin(n+k)(2\pi-x)$ and integrate from 0 to 2π, and we get, since term-by-term integration is allowable,

$$\int_0^{2\pi} (Ax+B) \sin(n+k)(2\pi-x)\, dx = -(n+k)^{-2} a_n \pi \sin 2k\pi,$$

whence

$$\int_0^{2\pi} (Ax+B)\, d\cos(n+k)(2\pi-x) = (n+k)^{-1} a_n \pi \sin 2k\pi,$$

that is, integrating by parts,

$$\Big[(Ax+B)\cos(n+k)(2\pi-x)\Big]_0^{2\pi} - \int_0^{2\pi} A \cos(n+k)(2\pi-x)\, dx = (n+k)^{-1} a_n \pi \sin 2k\pi,$$

that is,

$$(2\pi A+\beta) - B\cos 2k\pi + (n+k)^{-1} A \sin 2k\pi = (n+k)^{-1} a_n \pi \sin 2k\pi.$$

* Hobson's *Theory of Functions of a Real Variable*, p. 211.

Now let $n \to \infty$, then, since $A \sin 2k\pi$ is finite, and $a_n \to 0$, we get

$$(2\pi A + B) - B \cos 2k\pi = 0.$$

Therefore also, by the preceding equation,

$$A \sin 2k\pi = a_n \pi \sin 2k\pi.$$

Now again letting $n \to \infty$, we get

$$A \sin 2k\pi = 0;$$

and therefore

$$A = 0.$$

Hence also from the former equation

$$B = 0.$$

Thus, by the above expression for $(n+k)^{-1} a_n \pi \sin 2k\pi$, we have, for all values of n,

$$a_n = 0,$$

which proves the theorem.

22. The considerations above exposed lead naturally, as in the case of harmonic series,* from the theory of non-harmonic Fourier series to that of restricted non-harmonic Fourier series. The definition is as follows:

The p-th derived series of a non-harmonic Fourier cosine or sine series of $F(x)$ is said to be a non-harmonic restricted Fourier cosine or sine series of the p-th class, and to be restricted to one or more intervals (α, β), if, throughout each completely open interval in (α, β), $F(x)$ is a p-th integral. The p-th differential coefficient of $F(x)$ is then said to be the associated function of the restricted Fourier series in any of these intervals of restriction.

If, in addition, the coefficients of the restricted Fourier series tend to zero as we advance along the series, it is said to be "ordinary," and is denoted briefly as a non-harmonic R. F. series.

We have already seen that a non-harmonic Fourier sine or cosine series satisfies both these conditions in the closed interval $(0, 2\pi)$.

One of the fundamental properties of non-harmonic restricted Fourier

* W. H. Young, "On the Convergence of the Derived Series of Fourier Series" (1916), *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1919), pp. 195–236; "On the Ordinary Convergence of Restricted Fourier Series" (1917), *Proc. Roy. Soc.*, (A), Vol. 93, pp. 276–292; "On Restricted Fourier Series and the Convergence of Power Series" (1917), *Proc. London Math. Soc.*, Ser. 2, Vol. 17 (1919), pp. 353–366.

series* is embodied in the following theorems:—

THEOREM.—*The upper and lower functions at any point internal to the interval of restriction are independent of the nature of the associated function, except in an arbitrary small neighbourhood surrounding the point, provided the individual terms of the series tend to zero at the point as we advance along the series; indeed the part of the expression for the n-th partial summation depending on the remaining values of the function tends uniformly to zero.*

THEOREM.—*In the case when the typical coefficient itself converges to zero so that the series is "ordinary", we can go still further and assert that the series behaves at any internal point of its interval of restriction precisely like the Fourier series of a function equal to the associated function of the series in the neighbourhood of the point and having any convenient values elsewhere.*

These theorems remain true at the end-points, provided the associated function satisfy at the point in question the appropriate "continuation" formula applying to the point.

23. For the proof of these theorems we require a modified form of the expressions (A′) and (B′). These expressions involve only the values of $f(x)$ in the interval $(0, 2\pi)$. We shall now suppose that $f(x)$ is "continued" according to the appropriate law. The formulæ (A″) and (A‴) obtained are identical for the sine and cosine series, only varying in these "continuation" formulæ, and in the meaning of the symbol $A_n^{(k)}$. Taking, for example, the formula (A′) with the corresponding "continuation" formulæ

$$f(t) = f(-t),$$

$$\tfrac{1}{2}\{f(2\pi+t)+f(2\pi-t)\} = \cos 2k\pi \,.\, f(t),$$

we may transform (A′) as follows:

Writing in the first term of the bracket

$$x+t = 2\pi-u,$$

and in the second term $\quad x-t = u-2\pi,$

---

* These theorems were given by me in the communication to the Royal Society quoted above. The proof is reproduced here, slightly adjusted so as to agree with the present point of view, and so as to include the end-points.

we get for (A'), omitting the factor $1/(2\pi \sin 2k\pi)$,

$$\int_{-x}^{2\pi-x} f(x+u) \sin k(2\pi-u) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du$$
$$-\int_{x}^{2\pi+x} f(u-x) \sin k(u-2\pi) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du.$$

Adding and subtracting the integrals requisite to change the limits of integration in both these terms to $(0, 2\pi)$, and then making the obvious changes of the independent variable in the superfluous integrals, we get

$$\int_0^{2\pi} \{f(x+u)+f(u-x)\} \sin k(2\pi-u) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du$$
$$+\int_0^{x} [f(x-u) \sin k(2\pi+u) - f(u-x) \sin k(2\pi-u)$$
$$-\{f(x-u+2\pi)+f(2\pi+u-x)\} \sin ku] \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du.$$

Since f satisfies the "continuation" formulæ appropriate to a cosine series, the superfluous integral will be zero, and we get

$$s_n = \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \tfrac{1}{2} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \operatorname{cosec} \tfrac{1}{2}u \sin(n+\tfrac{1}{2})\, u\, du \qquad (A'')$$
$$= \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \tfrac{1}{2} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du$$
$$+ \frac{1}{\pi \sin 2k\pi} \int_0^{2\pi} \tfrac{1}{2} \{f(x+u)+f(x-u)\} \sin k(2\pi-u) \cos nu\, du.$$

The last of these integrals, being one half the sum of the n-th and the $-n$)-th coefficients in the non-harmonic Fourier cosine series of $\frac{1}{2}\{f(x+u)+f(x-u)\}$ is, as will be shewn in the next article, equal to $\frac{1}{2}\{A_n^{(k)}+A_{-n}^{(k)}\}$. Thus the partial summation s_n of our series is given by

$$s_n = \sum_{r=-n}^{n} A_r^{(k)} = \sum_{r=-n}^{n} a_r \cos(r+k)\, x$$
$$= \tfrac{1}{2}\{A_n^{(k)}+A_{-n}^{(k)}\} + \frac{1}{\pi \sin k\pi} \int_0^{2\pi} \tfrac{1}{2}\{f(x+u)+f(x-u)\}$$
$$\times \sin k(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du, \qquad (A''')$$

in which formula $\{f(x+u)+f(x-u)\}$ must be supposed to obey the "continuation" laws appropriate to the cosine series, since $\sum_{r=-\infty}^{\infty} A_r^{(k)}$ is a cosine series. If this series is a sine series the same formula holds, but the continuation is that appropriate in this case (15) and (16).

24. Indeed, supposing $f(t) = f(-t)$, we have

$$\int_0^{2\pi} \tfrac{1}{2}\{f(x+u)+f(x-u)\} \sin(n+k)(2\pi-u)\,du$$

$$= \int_x^{2\pi+x} \tfrac{1}{2}f(t)\sin(n+k)(2\pi-t+x)\,dt + \int_{-x}^{2\pi-x} \tfrac{1}{2}f(t)\sin(n+k)(2\pi-t-x)\,dt$$

$$= \int_0^{2\pi} f(t)\sin(n+k)(2\pi-t)\cos(n+k)x\,dt$$

$$-\int_0^{x} \tfrac{1}{2}\{f(t)\sin(n+k)(2\pi-t+x) - f(t)\sin(n+k)(2\pi+t-x)$$
$$-f(2\pi+t)\sin(n+k)(x-t) + f(2\pi-t)\sin(n+k)(t-x)\}\,dt$$

$$= \cos(n+k)x \int_0^{2\pi} f(t)\sin(n+k)(2\pi-t)\,dt, \qquad (19)$$

supposing the "continuation" formula (4) to hold.

If on the other hand the "continuation" formulæ (15) and (16) holds, we get, similarly,

$$\int_0^{2\pi} \tfrac{1}{2}\{f(x+u)+f(x-u)\} \sin(n+k)(2\pi-u)\,du$$
$$= \sin(n+k)x \int_0^{2\pi} f(t)\cos(n+k)(2\pi-t)\,dt. \qquad (20)$$

Thus supposing known the non-harmonic Fourier $\left\{\begin{matrix}\text{cosine}\\ \text{sine}\end{matrix}\right.$ series of $f(x)$, say

$$f(x) \sim \sum_{n=-\infty}^{\infty} A_n^{(k)} = \sum_{n=-\infty}^{\infty} a_n \begin{matrix}\cos\\ \sin\end{matrix} (n+k)x,$$

we have $$\tfrac{1}{2}\{f(x+u)+f(x-u)\} \sim \sum_{n=-\infty}^{\infty} A_n^{(k)} \cos(n+k)u. \qquad (21)$$

25. We proceed now to the proof of the theorems enunciated in § 22. Writing for the second integrated series,

$$G(x) \sim -\sum_{n=-\infty}^{\infty} (n+k)^{-2} A_n^{(k)} = -\sum_{n=-\infty}^{\infty} (n+k)^{-2} a_n \begin{matrix}\sin\\ \cos\end{matrix} (n+k)x,$$

we have seen that, by (A'''), and § 25,

$$-\sum_{r=-n}^{n} (r+k)^{-2} A_r^{(k)} = -\tfrac{1}{2}(n+k)^{-2}\{A_n^{(k)} + A_{-n}^{(k)}\}$$
$$+\frac{1}{2\pi \sin 2k\pi}\int_0^{2\pi} \{G(x+u)+G(x-u)\}\sin k(2\pi-u)\cot\tfrac{1}{2}u \sin nu\,du.$$

Thus the upper and lower functions of the non-harmonic restricted Fourier series in question, are the upper and lower limits, when $n \to \infty$, of the second differential coefficient of this expression. Since $A_n^{(k)} + A_{-n}^{(k)}$ tends by hypothesis to zero, we have only to consider the upper and lower limits of

$$I \equiv \frac{d^2}{dx^2} \int_0^{2\pi} \{G(x+u)+G(x-u)\} \sin k(2\pi-u) \cot \tfrac{1}{2}u \sin nu\, du.$$

In treating this expression, the first step is to introduce in place of

$$\sin k(2\pi-u) \cot \tfrac{1}{2}u,$$

a function $\phi(u)$, defined equal to it in $(e < u \leqslant 2\pi)$, and having at the origin the value $-2k$, while its differential coefficient is zero there. In $(0, e)$ we suppose $\phi(u)$ to be the integral of an integral, with the same value and the same left and right hand differential coefficients at the point e, and to be otherwise subject to no conditions. The function $[\phi(u)+2k]$ then satisfies the conditions imposed on $\lambda(u)$ in the extension of Riemann's Third Theorem (§ 19).

We thus have

$$\frac{d^2}{dx^2} \int_0^{2\pi} \{G(x+u)+G(x-u)\} \{\phi(u)+2k\} \sin nu\, du \to 0 \quad (n \to \infty).$$

Hence we may replace (I) in our investigation by

$$\frac{d^2}{dx^2} \int_0^{2\pi} \{G(x+u)+G(x-u)\} \{\sin k(2\pi-u) \cot \tfrac{1}{2}u - \phi(u) - 2k\} \sin nu\, du$$

$$= \frac{d^2}{dx^2} \int_0^{e} \{G(x+u)+G(x-u)\} \{\sin k(2\pi-u) \cot \tfrac{1}{2}u - \phi(u)\} \sin nu\, du$$

$$-2k \frac{d^2}{dx^2} \int_0^{2\pi} \{G(x+u)+G(x-u)\} \sin nu\, du.$$

We see then we only need to prove that the last term tends to zero.

For this purpose we introduce a new function $\psi(u)$, defined as equal to

$$1-\cos k(2\pi-u) \quad (e \leqslant u \leqslant 2\pi),$$

and having at the origin zero for its value and for that of its differential coefficient. This function when suitably filled in from 0 to e, satisfies the conditions imposed on $\lambda(u)$ in the extended form of Riemann's Third Theorem. Therefore, omitting for the moment the factor $-2k$, we may

replace the last term in question by

$$\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\,[1-\psi(u)]\sin nu\,du$$

$$=\frac{d^2}{dx^2}\int_0^{e}\{G(x+u)+G(x-u)\}\,\{1-\psi(u)-\cos k(2\pi-u)\}\,\sin nu\,du$$

$$+\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\cos k(2\pi-u)\sin nu\,du.$$

The last term is equal to

$$\frac{d^2}{dx^2}\int_0^{2\pi}\tfrac{1}{2}\{G(x+u)+G(x-u)\}\,\{\sin(k+n)(2\pi-u)-\sin(k-n)(2\pi-u)\}\,du$$

$$=\pi\sin 2k\pi\,\{A_n^{(k)}-A_{-n}^{k}\}\to 0.$$

Thus, finally, writing now $\phi(u)$ instead of

$$\phi(u)+2k\{1-\psi(u)-\cos k(2\pi-u)\},$$

we see that the partial summation s_n of our non-harmonic restricted Fourier series is given by the formula

$$s_n=o(1)+\frac{d^2}{dx^2}\int_0^{e}\tfrac{1}{2}\{G(x+u)+G(x-u)\}$$

$$\times\{\sin k(2\pi-u)\cot\tfrac{1}{2}u-\phi(u)\}\sin nu\,du/\pi\sin 2k\pi, \quad (22)$$

where $\phi(u)$ is the integral of an integral, with the value $-2k\cos 2k\pi$ at the origin, and $\sin k(2\pi-e)\cot\frac{1}{2}e$ at the point e. If the point x is internal to the interval $(0, 2\pi)$, we can, in this formula, choose e so small that no "continuation" is necessary. But if $x=0$, we have to regard G as an even function, and, if $x=2\pi$, we have to assume that the "continuation" formula used is that appropriate to a $\begin{cases}\text{cosine}\\ \text{sine}\end{cases}$ series.

Since our upper and lower functions are the upper and lower limits of s_n, this proves the first theorem of § 22.

25. Now if in the interval $(x-e, x+e)$ the function G is the integral of the integral of f, we may in the expression (22) perform the differentiation under the integral sign, writing f in place of G. By the theorem of Riemann-Lebesgue, the part of the integral involving the arbitrary function ϕ then tends to zero. Thus we are left with

$$s_n=o(1)+\frac{1}{\pi}\int_0^{e}\{f(x+u)+f(x-u)\}\,\frac{\sin k(2\pi-u)}{\sin 2k\pi}\cot\tfrac{1}{2}u\sin nu\,du.$$

Now, when $u \to 0$,

$$\left\{\frac{\sin k(2\pi u)}{\sin 2k\pi} - 1\right\} \cot \tfrac{1}{2}u = \frac{[\sin k(2\pi - u) - \sin 2k\pi]\cos \tfrac{1}{2}u}{\sin 2k\pi \sin \tfrac{1}{2}u} \to -\frac{k \cot 2k\pi}{\tfrac{1}{2}},$$

and is therefore bounded in the neighbourhood of the origin.

By the theorem of Riemann-Lebesgue, therefore, we may replace our expression by

$$s_n = o(1) + \int_0^e \{f(x+u) + f(x-u)\} \cot \tfrac{1}{2}u \sin nu \, du,$$

which is the Fourier expression.

This completes the proof of the second of the theorems of § 22.

26. We come now to the application of the results of the present paper to the theory of Bessel series. In particular we are now able to obtain the conditions of convergence at the end-point $z = 1$ of the interval of restriction of the Bessel series.

The asymptotic approximation

$$\sqrt{(\tfrac{1}{2}\pi x)}\, J_m(x) = \cos\psi + \frac{m^2 - \frac{1}{4}}{2x} \sin\psi - \frac{(m^2 - \frac{1}{4})(m^2 - \frac{9}{4})}{2!\,(2x)^2}\left[\cos\psi + \theta \frac{m^2 - \frac{16}{4}}{3\,(2x)}\right],$$

where $|\theta| < 1$ and $\psi = -x + \frac{1}{4}(2m+1)\pi$, holds up to *and including* the point $x = 1$.

Consequently we may regard as valid up to *and including* $z = 1$, the formal restrictions (21) and (29) obtained in my paper on "Series of Bessel Functions", viz., firstly, the relation (21),

$$\sum_{r=0}^{\infty} A_r J_m(k_r z) \equiv z^{-\frac{1}{2}} \{S_1 + z^{-1} S_2 + z^{-2} S_3\},$$

where S_3, and the first integrated series of S_2, and the second integrated series of S_1, are convergent series whose sums are integrals; while the second integrated series of the Bessel series on the left of this formal identity is a uniformly convergent series whose sum is the integral of an integral; and, secondly, the relation (29),

$$S_1 \equiv T + \beta y T_1 + T_2, \quad y = \pi(1 - z),$$

where T_2 and the first integrated series of T_1 converge uniformly to integrals, while

$$T = \sum_{r=0}^{\infty} (-)^{r+s} A_r \sqrt{\left(\frac{2}{\pi k_r}\right)} \cos(k+r)\, y.$$

We recall also the fact that, apart from an irrelevant factor, S_2 and T_1 are the first integrated series of S_1 and T respectively.

Now, by hypothesis $\quad A_r k_r^{-\frac{1}{2}} \to 0 \quad (r \to \infty)$,

and, integrating twice with respect to $y = \pi(1-z)$, the series

$$S_1 - \beta y T_1 - T_2,$$

we get the integral of an integral. Thus T *is a non-harmonic restricted Fourier series in* $(0 \leqslant y < 1)$, and therefore in this interval T_1, the first integrated series of T, converges to an integral, the convergence being uniform in every closed interval inside $(0 \leqslant y < 1)$. Similarly now we see that S_2 converges uniformly to an integral.

Thus we get from the given relations

$$\sum_{r=0}^{\infty} A_r J_m(k_r z) = z^{-\frac{1}{2}} \{T + \phi(z)\},$$

where $\phi(z)$ is an integral in $(0 < z \leqslant 1)$.

Hence, denoting the function associated with the Bessel series by $f(z)$, and that associated with the series T by $g(y)$, where $y = \pi(1-z)$, we get, after multiplying by $z^{\frac{1}{2}}$ and integrating twice with respect to $z = 1 - y/\pi$,

$$\int dz \int z^{\frac{1}{2}} f(z)\, dz = \int dz \int g\{\pi(1-z)\}\, dz + \int dz \int \phi(z)\, dz,$$

whence, differentiating twice, and choosing the function g suitably at the set of content zero at which the differentiation does not work,

$$g\{\pi(1-z)\} = z^{\frac{1}{2}} f(z) - \phi(z).$$

Thus, by our theory, the non-harmonic restricted Fourier series T behaves at the point $\pi(1-z)$ like the Fourier series of a function which in the neighbourhood of the point considered has the form

$$z^{\frac{1}{2}} f(z) - \phi(z),$$

provided the point z is *internal* to the interval (0, 1); and at the point $z = 1$ [which is the same as $\pi(1-z) = 0$], the function has this form, and on the right is reflected in the point $z = 1$. This will be satisfied if $z^{\frac{1}{2}} f(z)$ and $\phi(z)$ separately have this "continuation" property. Now $\phi(z)$ is an integral up to and including the point $z = 1$, and is therefore, when "continued" by reflection in this point, still an integral in the whole neighbourhood considered. Therefore its Fourier series converges uniformly, and, consequently, the term $\phi(z)$ may be suppressed in the expression for our auxiliary function. Also the Fourier series of $z^{\frac{1}{2}} f(z)$

behaves in $(0 < z < 1)$ like the Fourier series of $f(z)$, and the Fourier series of the function which in the neighbourhood of $z = 1$ is equal to $z^{\frac{1}{2}}f(z)$ on the left and is reflected on the right, behaves like that of the function equal to $f(z)$ on the left and reflected on the right, for in each case the factor removed represents a bounded function.

Thus, finally, observing that, in virtue of the relations found, the Bessel series behaves in respect of convergence, and so forth, according to the behaviour of the series T, we see that *the Bessel series converges, oscillates, or diverges, under precisely the same conditions and in precisely the same manner (uniform, bounded, &c.) as the Fourier series of a function which in a neighbourhood enclosing the point z considered is equal to $f(z)$, if $(0 < z < 1)$, and, if $z = 1$, has this form on the left of the point and is reflected in the point.*

A SIMPLE CONDITION FOR CO-APOLAR TRIANGLES

By Prof. William P. Milne.

[Received November 7th, 1918.—Read November 14th, 1918.]

It is known that, if ABC and DEF be any two triangles in the same plane, the locus of the point P moving so that the pencils $P[ABC]$ and $P[DEF]$ are apolar to one another is a cubic curve, which we call the Apolar Locus of ABC and DEF. The apolar locus is apolar to each of the triangles ABC and DEF regarded as degenerate class-cubics, and we call such types of triangles "apolar triangles." The converse problem may be stated as follows :—*Given a cubic curve, to find two apolar triangles which subtend at every point on the curve pencils of lines apolar to one another.* Such pairs of triangles are called "co-apolar triangles." In the *Proc. London Math. Soc.*, Ser. 2, Vol. 17, p. 237, I found the necessary and sufficient condition for co-apolarity in a perfectly symmetrical form by projecting the given pair of apolar triangles through a suitable point on the cubic into two new apolar triangles lying on the curve and having their six vertices on the same conic. By this means I was able to express the required condition in terms of the invariant properties of two triads of points on a conic. The present solution is more direct and less complicated than the above.

We shall deal with the reciprocal form of the above problem, because in the reciprocal form the geometry involved is more easy to visualize.

Let the given class-cubic Γ be inscribed in the two triangles S_1 and S_2, and let it be apolar to each of them. Let S_1 be chosen as the triangle of reference and let the three equations to S_1, S_2, Γ be chosen as follows

$$S_1 \equiv xyz = 0,$$

$$\begin{aligned} S_2 &\equiv (l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)(l_3x+m_3y+n_3z) \\ &\equiv ax^3+by^3+cz^3+3a_2x^2y+3a_3x^2z+3b_3y^2z+3b_1y^2x+3c_1z^2x \\ &\qquad +3c_2z^2y+6kxyz = 0, \end{aligned}$$

$$\Gamma \equiv A_2l^2m+A_3l^2n+B_3m^2n+B_1m^2l+C_1n^2l+C_2n^2m = 0.$$

If, now, S a cubic curve represented by the general equation be taken and the apolar envelope of S and S_1 be formed, and if this apolar envelope be identified with Γ, we shall find that the whole system of cubic-loci co-apolar with S_1, with respect to Γ, is:

$$(\lambda x^3+\mu y^3+\nu z^3+6\rho xyz)$$
$$+(C_2x^2y-B_3x^2z+A_3y^2z-C_1y^2x+B_1z^2x-A_2z^2y)=0, \quad (1)$$

where λ, μ, ν, ρ may have any values, and the contents of the second bracket represent the apolar locus of the triangle S_1 regarded as an envelope and Γ. Now, if the triangle S_2 regarded as a cubic curve composed of three straight lines be one of the members of the system (1), we see that one of the pencil of cubics passing through the nine points of intersection of the sides of the triangle S_2 with the apolar locus of S_1 and Γ, possesses S_1 as an inflexional triangle in virtue of the form

$$\lambda x^3+\mu y^3+\nu z^3+6\rho xyz.$$

We therefore have the following result:

If two apolar triangles be co-apolar with respect to a class-cubic, the sides of either triangle intersect the apolar locus of the other triangle and the class-cubic in nine points, through which passes a cubic curve having that other triangle as an inflexional triangle.

We shall now prove the converse theorem, viz.

The necessary and sufficient condition that two apolar triangles shall be co-apolar with respect to a class-cubic, is that the sides of either triangle must intersect the apolar locus of the other triangle and the class-cubic in nine points, through which passes a cubic curve having that other triangle as an inflexional triangle.

To prove the above, let us form the general equation to the pencil of class-cubics inscribed in the triangles S_1, S_2, and apolar to them (*i.e.* satisfying eight independent linear conditions).

One such class-cubic is the apolar envelope of S_1 and S_2, viz.

$$c_2l^2m-b_3l^2n+a_3m^2n-c_1m^2l+b_1n^2l-a_2n^2m=0. \quad (2)$$

Another class-cubic satisfying the above eight conditions is easily seen to be

$$\Sigma l_1l_2l_3(mn_1-nm_1)(mn_2-nm_2)(mn_3-nm_3)=0;$$

which, in accordance with the notation for S_2 given above, can be thrown into the form

$$cb_1l^2m-bc_1l^2n+ac_2m^2n-ca_2m^2l+ba_3n^2l-ab_3n^2m=0. \quad (3)$$

The general equation to the required pencil of class-cubics is therefore

$$(\lambda' cb_1+\mu' c_2)\, l^2m-(\lambda' bc_1+\mu' b_3)\, l^2n+(\lambda' ac_2+\mu' a_3)\, m^2n-(\lambda' ca_2+\mu' c_1)m^2l$$
$$+(\lambda' ba_3+\mu' b_1)\, n^2l-(\lambda' ab_3+\mu' a_2)\, n^2m = 0. \quad (4)$$

The apolar locus of S_1 and (4) is, therefore,

$$(\lambda' ab_3+\mu' a_2)\, x^2y+(\lambda' ac_2+\mu' a_3)\, x^2z+(\lambda' bc_1+\mu' b_3)\, y^2z+(\lambda' ba_3+\mu' b_1)\, y^2x$$
$$+(\lambda' ca_2+\mu' c_1)\, z^2x+(\lambda' cb_1+\mu' c_2)\, z^2y = 0. \quad (5)$$

Now, of all the members of the pencil of cubic curves passing through the nine points of intersection of the triangle S_2 and the curve (5), in general no one possesses S_1 as an inflexional triangle, because no member has the form

$$\lambda x^3+\mu y^3+\nu z^3+6\rho xyz = 0.$$

If, however, λ' vanish in (4) and (5), *one* member of the pencil defined by the intersections of S_2 and (5) does possess S_1 as an inflexional triangle. If λ' vanish in (4), we obtain the apolar envelope of S_1 and S_2. Hence, of all the members of the pencil of class-cubics (4), each of which is inscribed to the triangles S_1, S_2 and apolar to them, one and only one member possesses the property that the sides of either triangle intersect the apolar locus of the other triangle and that particular member in nine points, through which passes a cubic curve having that other triangle as an inflexional triangle.

The condition under investigation is, therefore, both necessary and sufficient.

ON A FORMULA FOR AN AREA

By W. H. Young.

[Received March 28th, 1919.—Read April 24th, 1919.]

Part I.—*Introduction.*

1. In the treatment of the area of a curve, it has been usual to suppose the curve to be a simple Jordan curve, in other words, a curve defined by a continuous (1, 1) correspondence with a straight line or circle. Such a curve, when closed, divides the plane into two distinct parts, the "inside" and "outside" of the curve, the former of which is a simply connected region. Such a curve can be approximated to by polygons, without loops, internal and external to the curve, and the area is defined as the common unique limit, whenever this exists (as it will if the curve is rectifiable), of the area of these polygons; and this area is the same as the content of the set of points inside the curve, together with, or omitting, the points of the curve itself, as we please.*

The idea, however, that an area, when given by a formula, is not always to be conceived of in this sense, is a familiar one. Thus, for example, the formula

$$A = \tfrac{1}{2}\int \{x\,dy - y\,dx\},$$

which in the case of a simple closed Jordan curve gives its area, when it exists, leads, when the integral is taken round a curve in which there are loops, only to the sum of the areas of those loops, when the area is regarded as a directed quantity (*directed area*), got by attaching an appropriate sign to the content of the inside of the loop, this sign being + when a point, describing the curve in the manner prescribed by the equations defining it, has the inside of the loop on its left (anticlockwise motion), and − when the inside of the loop is on the right (clockwise

* All these results are discussed in Chapters x and xiii of the *Theory of Sets of Points*, and references given.

motion). Thus in the case of the lemniscate, we are led to the value zero.

2. In the theory of the transformation of the variables in a multiple integral, based though it frequently is on geometrical representation, no application, as far as I know, of the idea of *directed area* is to be found. This is no doubt connected with the fact that it has been usual to regard the independent variables as always preserving their own sign, and either steadily increasing, or steadily decreasing. The moment, however, we introduce integration with respect to a function of bounded variation, this peculiarity in the process of integration disappears. The function of bounded variation will not, in general, be monotone, and may change sign. This fact may be of importance, even in one dimension. Thus in the method of integration by substitution, it is worth while to state the fundamental theorem in the case of a single variable in such a manner as to permit of the old variable x being a not necessarily monotone function of the new one t. This corresponds in the theory of lengths to the mode of estimation which consists in regarding journeys over the same ground in opposite directions as cancelling one another. The formula which lies at the basis of the theory is

$$\int dx = x = \int \frac{\partial x}{\partial t}\, dt,$$

and, as is well known, is true if x is any (absolutely convergent) integral with respect to t, and $\partial x/\partial t$ denote any one of its derivates.*

When we come to two dimensions, the question becomes at once more difficult and more interesting. Under what conditions, we ask, may we, in a multiple integral taken with respect to x and y, substitute for x and y functions of two new variables u and v?

Just as in one dimension the problem we have to solve is in its simplest form the determination of a formula for a directed length as the integral (or sum) of directed lengths, so, in our case, it is the problem of determining a formula for a directed area as the integral, *i.e.* the limit of the algebraic sum, of directed areas.

The problem is a more difficult one, not only owing to the comparative unfamiliarity of the concept of "directed area", but also because the analysis required involves considerations of a more subtle character.

* H. Lebesgue, *Leçons sur l'intégration*. See, for instance, my paper on "Functions of Bounded Variation," *Quarterly Journal*, Vol. 42 (1910), p. 81.

The directed area formula itself, on the other hand, is well-known: it is

$$A = \iint \frac{\partial(x, y)}{\partial(u, v)} du\, dv,$$

where $\frac{\partial(x, y)}{\partial(u, v)}$ denotes the Jacobian

$$\begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Here $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial x}{\partial v}$ and $\frac{\partial y}{\partial v}$ are in the first instance, and in the cases usually considered, the actual partial differential coefficients of the functions x and y with respect to u and v. But they may evidently without any inconvenience be regarded as any partial derivates, provided only they exist as differential coefficients, except at a set of plane content zero of values (u, v). What is required then is to establish this formula under conditions which have no reference to the sign of the Jacobian, or even to the signs of the partial derivates, a suitable definition having been previously devised for *area*.

3. At this point, however, it may be well to deal with the question which may already have suggested itself to the reader, as to the value of such considerations. It might be sufficient to reply generally by adducing merely the advantage of possessing a more general form for so fundamental a formula as that which constitutes the expression of the rule for the transformation of the variables in a multiple integral. The fewer the restrictions imposed on the functions with which it is concerned, the greater the potential usefulness of the formula. It is a great advantage not to have to prove *a priori* that such conditions are satisfied, even when in point of fact, *a posteriori*, this turns out to be the case. Moreover the greater generality of the statement of a theorem will often involve the necessity for devising a new method for proving it, and this may be actually not only more powerful, but more fruitful than that which presents itself more naturally in the simpler case. This occurs in the present instance; for the method to which I have been led enables us, even in the simpler case, to dispense with a number of useless restrictions, which

are seen to have their origin merely in the imperfections of the older method of proving the formula.

The mere fact that in one dimension these considerations are fundamental, is of itself sufficient reason for attempting to generalise them. I may say, however, that it is in the progress of researches into the area of curved surfaces that the necessity for modifying the usual definition of the area of a plane curve has been forced on my mind. It has been the modification which I have made in that concept, and the introduction of the analogous concept for *the area of a skew curve*, that have enabled me to overcome the difficulties in this theory which seem to have baffled all previous investigators. It has thus been rendered possible to evolve a simple and harmonious treatment not only of the concept of the area of a curved surface, but also of the analogous concepts in the case of curved manifolds of any number of dimensions in space of any number of dimensions.

4. To return to the generalisation of the concept of the *area of a closed curve*, it is clear that this should be such that it reduces in the particular case when the curve is a simple Jordan curve to the usual definition. As a particularly simple formula for the area of such a Jordan curve is known to us, it is open to us to attempt either to generalise the definition itself, or to devise a new definition to which this simple formula is still applicable. The latter will naturally suggest itself, if practicable; for it is desirable that in any case this formula should continue to hold good.

We are thus at once led to the following definition, which, it will be seen, is applicable to every closed curve for which the formula in question has a meaning, and accordingly certainly whenever the curve is rectifiable, *i.e.* has a length in Jordan's sense, or more generally if, the curve being defined by the equations

$$x = x(u), \quad y = y(u), \quad (a \leqslant u \leqslant b),$$

at least one of the continuous functions $x(u)$ and $y(u)$ has bounded variation.*

* The necessary and sufficient condition that the curve should be rectifiable is, as Jordan showed, that both $x(u)$ and $y(u)$ should be, not only continuous, but of bounded variation. If one of these functions at least has bounded variation, both being continuous, the plane content of the curve is easily shown to be zero.

Definition.

Inscribe in the curve, which we shall now suppose closed, any polygon in the usual way, so as, when the curve is rectifiable, to form an approximation to the length of the curve. Imagine vectors to be represented in respect of magnitude, line of action, and sense, by the sides of this polygon, supposed described in the sense in which u increases. Take the moments of these vectors about any point O in the plane. Then it may be shewn that—

(i) *The algebraic sum of these moments is independent of the position of the point O;* it is, of course, the moment of the couple equivalent to the imagined vectors, regarded as forces, and is equal to twice the area of the polygon in the usual sense, when the polygonal line does not cut itself.

(ii) *The number expressing this sum has a unique limit $2A$, as the number of the sides of the polygon is indefinitely increased and their lengths indefinitely decreased,* so that the perimeter of the polygon has the length of the curve, if rectifiable, as unique limit.

(iii) *The formula holds,*

$$A = \tfrac{1}{2}\int_a^b \{x(u)dy(u) - y(u)dx(u)\}.$$

This quantity A is defined to be the area of the curve.

5. It is not necessary for the purposes of this paper to give any other definition, but it will be easily seen that the area so defined is equivalent to the following, which may be regarded as the generalisation of the usual definition of the area of a simple Jordan curve.

The polygonal line inscribed in our closed curve may be thought of as forming a number of loops, some described in a clockwise, and others in an anti-clockwise manner. The sum of the areas of these loops, taken with proper sign, is identical with half the sum of the moments used in our definition, and will accordingly lead to the same unique limit A. And it should be noted that, though there is a certain degree of arbitrariness in the choice of these loops, the figure taken as a whole is perfectly definite in each case, and divides the whole plane up into connected sets of points, which, from the point of view of area, have a definite multiplicity. Thus certain points will be counted so many times positively, others nega-

tively, while the rest will be regarded as external to the figure. As we proceed to the limit, certain of these sets will in essentials not change, just as in the gradual generation of the black intervals of a perfect set by means of increasing sets of intervals finite in number.

6. We are now able to state the main result of the paper:—

Taking first the case of two dimensions, let

$$x = x(u, v), \qquad y = y(u, v),$$

be functions of (u, v) *possessing the property of having all their partial derivates with respect to* u *and* v *bounded for all values of* (u, v) *in the fundamental rectangle* $(a, c\,;\ b, d)$, *that is*

$$a \leqslant u \leqslant c, \qquad b \leqslant v \leqslant d,$$

and let A *be the area of the curve in the* (x, y)*-plane which is the image of the perimeter of this rectangle.*

Then
$$A = \int_a^c du \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv,$$

where $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ *represent any of the partial derivates of* x *and* y *with respect to* u *and* v.

More generally the same is true, if the partial derivates are not bounded, provided only the following conditions are satisfied:—

(i) $x(u, v)$ *is an integral with respect to* u, *and an integral with respect to* v;

(ii) $y(u, v)$ *is an integral with respect to* u, *and an integral with respect to* v;

(iii) $\frac{\partial y}{\partial v}$ *and* $\frac{\partial x}{\partial v}$ *are, except for a set of values of* v *of content zero, numerically less than summable functions of* v *alone,* $\mu(v)$ *and* $M(v)$ *say;*

(iv) *the same as* (iii) *with* u *and* v *interchanged, or more generally* $\iint \frac{\partial x}{\partial u}\mu(v)\, du\, dv$, $\iint \frac{\partial y}{\partial u} M(v)\, du\, dv$, *exist as absolutely convergent integrals.*

It will have been noticed that the conditions in the theorem just given

are stated to be sufficient; there is no suggestion that they are necessary. But this is in accordance with what might be expected, as long, at least, as the conditions are expressed in terms of the quantities in the integrand. We know of no necessary and sufficient conditions in the case of the analogous problem in one dimension. Thus no knowledge of the nature of the derivate of a function constitutes a necessary as well as sufficient condition that its primitive function should be the integral of it. The analogy between the cases of one and two dimensions is not only of interest but it appears to belong to the nature of things: we are in presence, in fact, of a generalisation other than the usual one, of a function of a single variable, namely, of a functional relation between two planes which generalises for two dimensions the notion of a function of bounded variation of a single variable, and also that of integral. For the development of this idea I must refer my readers to a communication which I am making elsewhere.

7. I have not contented myself with proving these results in this general form, but I have begun by shewing how, by simpler methods, results, which, though less general than these, are none the less considerably more general than any hitherto formulated, may be obtained. I have hoped in this way to be of service to the worker in a double capacity, both as pioneer and guide, and this the more because I was actually led to the most general of my results by passing through the intermediate stages.

8. The expression for an n-dimensional volume which corresponds to that for an area in two dimensions, is formally an immediate generalisation of the latter. The actual proof of the more general formula, which involves for its proper comprehension a somewhat more developed analytical machinery, and a certain insight into the nature of the concept of surface, volume, and their generalisations, I have thought best to defer till after the publication of some of my results connected with the theory of these concepts. I have also not entered into the general question, referred to above, of the transformation of the variables in a multiple integral, connected as it is with the generalisation of the formula for, and of the concept of, a hyper-volume.

PART II.—*On the Notion of Area.*

10. As indicated in the introduction, we define the area of a curve as follows :—

Let the curve be defined by

$$x = x(u), \quad y = y(u),$$

where $x(u)$ and $y(u)$ are continuous one-valued functions of u, for values of u in the interval $(a \leqslant u \leqslant b)$, having the same values at a and b, viz.

$$x(a) = x(b), \quad y(a) = y(b),$$

so that the curve is *closed*, and may be called a *circuit*, or *contour*.

Divide the range of u up into any finite number of stretches, by means of dividing points $p_0 = a_1, p_1, p_2, \ldots, p_m = b$, each stretch being of length less than a certain norm e. Join the corresponding points

$$A, P_1, P_2, \ldots, P_{m-1}, A$$

on the curve to form an inscribed polygon.

Imagine forces, represented in magnitude, line of action, and sense, by the sides of this polygon in order, supposed determined in the sense of u increasing.

Denote by F the sum of the moments of these forces about any point in the plane, *i.e.* the moment of the resulting couple, counted positive when anti-clockwise in sense. Then, if, as the norm e tends to zero, so that $m \to \infty$, the number F has a unique limit $2A$, the limit A is called *the area of the curve.*

It is immediately evident that, with this definition, the area of the perimeter of a triangle is the area as defined in elementary geometry of the triangle, and that the area of an ordinary polygonal line, bounding a simply connected region is the area in the usual sense of that region.

THEOREM.—*If* $x = x(u)$, $y = y(u)$ *be any rectifiable* closed curve,† it necessarily possesses an area. Moreover this area is given by the formula*

$$A = \tfrac{1}{2}\int_a^b \{x(u)\,dy(u) - y(u)\,dx(u)\} \ldots,$$

---

* More generally, if *one* of the continuous functions $x(u)$ and $y(u)$ has bounded variation (see footnote to § 4), in which case also the expression (I) is perfectly defined.

† Jordan curve in the general sense, not necessarily *simple*.

the integrals being "Riemann" integrals, taken with respect to the functions of bounded variation $x(u)$ and $y(u)$.

Let x_0, y_0 be any fixed point in the plane of (x, y). Then the algebraic sum of the moments of the forces about (x_0, y_0) is equal to

$$F = \Sigma \begin{vmatrix} x_0, & x(u), & x(u+h) \\ y_0, & y(u), & y(u+h) \\ 1, & 1, & 1 \end{vmatrix} = \Sigma \begin{vmatrix} x(u)-x_0, & x(u+h)-x(u) \\ y(u)-y_0, & y(u+h)-y(u) \end{vmatrix},$$

where u is the coordinate appertaining to any one of the points of division $p_0, p_1, \ldots, p_{m-1}$, and $u+h$ that appertaining to the next point.

The coefficient of x_0 is

$$-\Sigma\{y(u+h)-y(u)\} = 0;$$

similarly that of y_0 is zero. Thus

$$\begin{aligned} F &= \Sigma\{y(u+h)x(u)-y(u)x(u+h)\} \\ &= \Sigma[\{y(u+h)-y(u)\}x(u)-\{x(u+h)-x(u)\}y(u)]. \end{aligned}$$

But, since the curve is rectifiable, the continuous functions $x(u)$ and $y(u)$ have bounded variation,* and therefore each has a "Riemann" integral with respect to the other. In other words,

$$\Sigma\{x(u+h)-x(u)\}y(u)$$

tends, when the norm e tends to zero, to a unique limit, and this is the Riemann integral

$$\int_a^b y(u)\,dx(u),$$

and a similar statement holds, with x and y interchanged. Therefore, as $e \to 0$,

$$F \to \int \{x(u)\,dy(u)-y(u)\,dx(u)\}.$$

This proves the theorem.

11. We now consider a correspondence between two planes, in which

* See footnote *supra*: it is only necessary that one of the functions shculd have bounded variation for the argument here.

the coordinates are respectively (x, y) and (u, v), defined by the equations

$$x = x(u, v), \quad y = y(u, v).$$

Denote by the term *fundamental* (u, v)*-rectangle*, the rectangle $(a, c\,;\ b, d)$, that is $(a \leqslant u \leqslant c)$, $(b \leqslant v \leqslant d)$ in the (u, v)-plane, throughout which the correspondence is supposed to hold.

Corresponding to this rectangle we shall obtain a certain configuration of points in the (x, y)-plane, differing very essentially in character, in general, from a rectangle; we shall call this configuration *the rectangle-image*, and denote it by S.

We shall not suppose the correspondence (1, 1) in the two planes, but we shall at once make the hypothesis that *the image of every stretch on a straight line, parallel to a side of the* (u, v)*-rectangle, and terminated on its periphery is a rectifiable* curve.* This is equivalent to the condition that $x(u, v)$ and $y(u, v)$ should both be continuous functions of bounded variation of u for each fixed value of v, and also continuous functions of bounded variation of v, for each fixed value of u.

With this understanding it follows at once from the previous article, that the curve in the (x, y)-plane which is the image of the perimeter of any rectangle in the (u, v)-plane, whose sides are parallel to the axes of u and v, has an area.

In the case of the perimeter of the fundamental (u, v)-rectangle, the area of the perimeter-image is moreover given by the formula

$$\begin{aligned} A = & \int_a^c \{x(u, b)\, dy(u, b) - y(u, b)\, dx(u, b)\} \\ & + \int_b^d \{x(c, v)\, dy(c, v) - y(c, v)\, dx(c, v)\} \\ & - \int_a^c \{x(u, d)\, dy(u, d) - y(u, d)\, dx(u, d)\} \\ & - \int_b^d \{x(a, v)\, dy(a, v) - y(a, v)\, dx(a, v)\}, \end{aligned} \tag{II}$$

which, if $x(u, v)$ and $y(u, v)$ are integrals with respect to v, when u is

* Or at least such a more general curve that the above formula holds, see previous footnotes.

constant, and integrals with respect to u, when v is constant, becomes*

$$A = \int_a^c \left\{ x(u, b) \frac{\partial y(u, b)}{\partial u} - x(u, d) \frac{\partial y(u, d)}{\partial u} - y(u, b) \frac{\partial x(u, b)}{\partial u} + y(u, d) \frac{\partial x(u, d)}{\partial u} \right\} du$$
$$+ \int_b^d \left\{ x(c, v) \frac{\partial y(c, v)}{\partial v} - x(a, v) \frac{\partial y(a, v)}{\partial v} - y(c, v) \frac{\partial x(c, v)}{\partial v} + y(a, v) \frac{\partial x(a, v)}{\partial v} \right\} dv. \quad \text{(II')}$$

12. Our object is to transform this expression for A into an integral of the form

$$A = \int_a^c du \int_b^d \frac{\partial(x, y)}{\partial(u, v)} du\, dv, \quad \text{(III)}$$

with the minimum of additional conditions which the circumstances permit.

If we make sufficient additional assumptions the transformation is almost immediate.

Thus, if we suppose that, not only $x(u, v)$ and $y(u, v)$, but also their partial derivates $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, are integrals with respect to v, and $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ integrals with respect to u, so that also $x \frac{\partial y}{\partial u}$ and $y \frac{\partial x}{\partial u}$ are integrals with respect to v, and $x \frac{\partial y}{\partial v}$ and $y \frac{\partial x}{\partial v}$ are integrals with respect to u, we may directly carry out the process of transformation of (II') into (III), by writing (II') in the form

$$A = \int_a^c du \int_b^d \frac{\partial}{\partial v} \left\{ y(u, v) \frac{\partial x(u, v)}{\partial u} - x(u, v) \frac{\partial y(u, v)}{\partial u} \right\} dv$$
$$- \int_b^d dv \int_a^c \frac{\partial}{\partial u} \left\{ y(u, v) \frac{\partial x(u, v)}{\partial v} - x(u, v) \frac{\partial y(u, v)}{\partial v} \right\} du,$$

* Readers of my memoir on "Integration with respect to a Function of Bounded Variation" in these *Proceedings*, Ser. 2, Vol. 13 (1914), pp. 111–150, should notice that the formula

$$\int_a^b f(x)\, dg(x) = \int_a^b f(x)\, g'(x)\, dx,$$

when $g(x)$ is an integral, though not expressly given there, follows easily, by the methods there explained, from the particular case in which $f(x)$ is a simple l or u function, in which case the formula is manifestly true.

which, when we differentiate out and cancel the terms that appear in each of the two integrals, reduces to (III).

We notice that this direct process of transformation fails, when the mixed repeated differential coefficients $\frac{\partial^2 y}{\partial u\,\partial v}$, $\frac{\partial^2 x}{\partial u\,\partial v}$, do not exist at every point of some set of positive content: even when they exist *almost everywhere*, the process fails if $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$ are not integrals with respect to v, and $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ integrals with respect to u.

13. We have accordingly to employ an indirect method, if we are to obtain results of anything like a general character.

For this purpose it is convenient to use the fundamental property of a system of forces, namely that it is not altered by the introduction of any number of pairs of equal opposite forces.

Imagine the fundamental (u, v)-rectangle divided up in any manner into a finite number of sub-rectangles, by parallels to the axes of u and v. Then divide each sub-rectangle into a pair of triangles by the diagonal which intercepts positive lengths on both axes (*i.e.* sloping down from left to right).

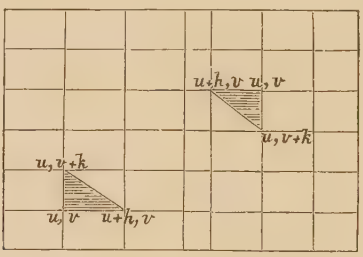


FIG. 1.

Denoting by (u, v) that vertex of such a triangle at which there is a right angle, and by $(u+h, v)$ and $(u, v+k)$ the other vertices, the area of the triangle is $\frac{1}{2}hk$. Adjoined to each such point (u, v) of division, there will then be two* pairs (h, k), in one of which h and k are both positive, and in the other both negative (Fig. 1).

We shall denote by $\bar{h}$ and $\bar{k}$ the greatest of the absolute values of all

* Exception of course made of the points (u, v) marked on the perimeter of the fundamental rectangle, which have each one pair (h, k). It will be readily seen that this does not affect our reasoning, and need not therefore be specially referred to in detail.

the h's and k's which occur in this division of the whole fundamental rectangle into sub-rectangles. The vertices of the triangles we shall for brevity refer to as the "marked points" at this stage.

14. Let the corresponding marked points in the (x, y)-plane in the same order as their correspondents,

$$(u, v), \quad (u+h, v), \quad (u, v+k)$$

be

$$(x, y), \quad (x+\Delta x, y+\Delta y), \quad (x+\Delta' x, y+\Delta' y).$$

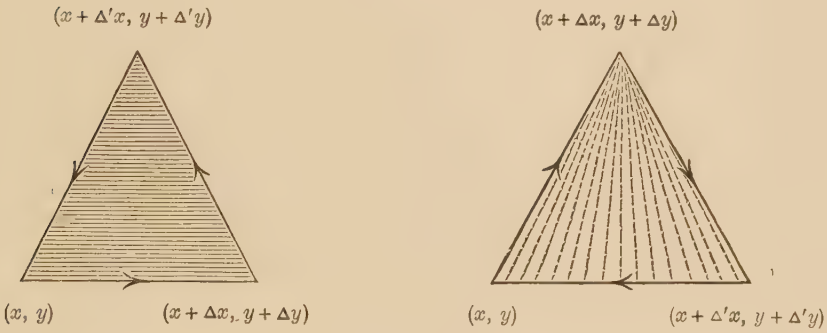


FIG. 2.

With the usual convention as to *sign*, the area being here a directed quantity, the area of the triangle in the (x, y)-plane having these vertices is

$$\tfrac{1}{2}\begin{vmatrix} x, & y, & 1 \\ x+\Delta x, & y+\Delta y, & 1 \\ x+\Delta' x, & y+\Delta' y, & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}(\Delta x\,\Delta' y - \Delta' x\,\Delta y) = \tfrac{1}{2}\{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\}$$
$$-\tfrac{1}{2}\{x(u, v+k)-x(u, v)\}\{y(u+h, v)-y(u, v)\},$$

and is positive or negative according as the order of the vertices so arranged is anticlockwise or clockwise. The area may even be zero, if the three points in the (x, y)-plane are collinear, or if two or more coincide.

15. It will be remarked that it is only the vertices (*marked points*) of the triangles in the two planes which "correspond", not the sides. It is convenient to speak of the two triangles as being "related". The "rela-

tion" is completely determined by, but not identical in general with, our "correspondence". It is only the *three-point*, so to speak, in the (x, y)-plane, and not *the three-side* which is the image of the "related" figure in the (u, v)-plane, images being determined by the correspondence

$$x = x(u, v), \quad y = y(u, v).$$

It is clear, however, that we can utilise the areas of these triangles for the purpose we have in view. In fact the sum of their areas, taken with proper sign, is precisely half the moment of the system of forces represented by the polygonal figure in the (x, y)-plane whose vertices correspond to the marked points in order on the perimeter of the fundamental (u, v)-rectangle. To see this we merely have to replace the directed area of each triangle, regarded as a couple, by forces along the sides, represented also in magnitude, line of action and sense by these sides, the sense being that given by the order of the vertices. Each side of such a triangle which is not "related" to a stretch on the perimeter of the fundamental (u, v)-rectangle, will occur as a common side of two triangles, as in the (u, v)-plane, and the two forces along it will necessarily be of opposite sense, and accordingly cut one another out.

We thus arrive at the conclusion that the area A for which we are seeking a second expression, this time as a double integral, is the limit of the area of the polygonal figure, when $\bar{h} \to 0$, $\bar{k} \to 0$.

16. Along the perimeter of the fundamental (u, v)-rectangle, x and y are functions of a single variable t, which for two of the sides of the rectangle is denoted by u, and the increment by h, and for the other two sides of the rectangle is denoted by v, and the increment by k.

It follows accordingly, by the theory of the "Riemann" integral, since A has been expressed by the sum of such integrals (Formula II), that it is immaterial whether we make the $\bar{h}$'s and $\bar{k}$'s approach zero simultaneously or consecutively. The sum of the areas of the triangles has accordingly necessarily a unique double limit, while it is open to us to calculate this double limit by any process by which $\bar{h}$ and $\bar{k}$ approach zero.

We shall find it convenient to make first one of these quantities tend to zero, and afterwards the other. We are thus able to deal with the separate terms which occur in the expression for the area of each triangle.

It is not indeed evident, and probably not true, that, on the hypotheses which we propose to make, the double limit is unique, unless all the terms due to all the triangles are taken into account at one and the same time.

17. We have thus to consider the summation of pairs of expressions, one member of each pair being of the type

$$\tfrac{1}{2}\{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\}$$
$$-\tfrac{1}{2}\{x(u, v+k)-x(u, v)\}\{y(u+h, v)-y(u, v)\},$$

where $h>0$, $k>0$, and the other of a similar type, except that then $h<0$, $k<0$.

We proceed to discuss the summation formed from one member of each pair separately, and write, with an obvious notation

$$F_1 = \tfrac{1}{2}\sum_a^c \sum_b^d \{x(u+h, v)-x(u, v)\}\{y(u, v+k)-y(u, v)\},$$
$$(h>0,\ k>0) \text{ or } (h<0,\ k<0).$$

It follows by the theory of the "Riemann" integral, as already utilised, that, equally whether h and k are always positive in the summation, or always negative, this expression tends, if we make $\bar{k}\to 0$, to

$$\tfrac{1}{2}\int_b^d \sum_a^c \{x(u+h, v)-x(u, v)\}\, dy(u, v).$$

Accordingly, treating the other terms in the same way, we get

$$A = \operatorname*{Lt}_{h\to 0} \tfrac{1}{2}\int_b^d \Big[\sum_a^c \{x(u+h, v)-x(u, v)\}\, dy(u, v)$$
$$-\sum_a^c \{y(u+h, v)-y(u, v)\}\, dx(u, v)\Big], \quad \text{(IV)}$$

where each (u, v) appears in two terms, one with $h>0$, and one with $h<0$.

18. We have now to make such assumptions as will enable us, in the result just obtained, actually to proceed to the limit with $\bar{h}$.

There is, however, one case so simple that it is best treated by means of the double summation.

THEOREM.—*If* $x(u, v)$ and $y(u, v)$ *are such functions that the partial differential coefficients* $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, *and* $\frac{\partial y}{\partial v}$ *everywhere exist and are continuous functions of* (u, v), *then the area* A *of the rectangle-image is given by the expression*

$$A = \int_a^c du \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$

By the Theorem of the Mean for Differential Coefficients, we may write

$$F_1 = \tfrac{1}{2} \sum_a^c \sum_b^d \frac{\partial x(u+\theta_1 h, v)}{\partial u} \frac{\partial y(u, v+\theta_2 k)}{\partial v} hk,$$

where $$|\theta_1| \leqslant 1, \quad |\theta_2| \leqslant 1.$$

Now we may suppose the division of the fundamental (u, v)-rectangle such that the oscillation of each of the partial differential coefficients $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ is less than ϵ [since these differential coefficients are continuous functions of (u, v)], in each of the sub-rectangles into which the fundamental (u, v)-rectangle has been divided.

Thus the expression may be written

$$F_1 = \tfrac{1}{2} \sum_a^c \sum_b^d \left\{\frac{\partial x}{\partial u} + \theta_3 \epsilon\right\} \left\{\frac{\partial y}{\partial v} + \theta_4 \epsilon\right\} hk.$$

Accordingly the whole area may be obtained by considering a summation of the form

$$\sum_a^c \sum_b^d \left\{\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}\right\} hk + \epsilon M,$$

where M is a bounded function of the various quantities concerned, and making all the h's and k's diminish without limit.

By the theory of the double integral, we then necessarily get a quantity which differs from the required double integral by a quantity of the type $\theta \epsilon B$, where $|\theta| \leqslant 1$, and B is a finite constant. As ϵ is as small as we please, this proves the required result.

19. We shall now, as in (II′), suppose that $x(u, v)$ *and* $y(u, v)$ *are each integrals with respect to* v *for each fixed value of* u, *and integrals with respect to* u *for each fixed value of* v.

Then our formula (IV) takes the form*

$$A = \operatorname*{Lt}_{h \to 0} \tfrac{1}{2} \int_b^d \sum_a^c \Big[\{x(u+h, V) - x(u, V)\} \frac{\partial y(u, V)}{\partial V} - \{y(u+h, V) - y(u, V)\} \frac{\partial x(u, V)}{\partial V} \Big] \partial V. \quad \text{(V)}$$

* In formulæ like the present involving (1) a variable with respect to which we sum, assuming a finite number of values at each stage, and therefore in all a countably infinite set of values, dense everywhere, and (2) a continuous variable with respect to which we integrate. I use small letters u, v for the former and capitals U, V for the latter. It may be remarked that, as regards the fixed values, the supposition made so far only applies to the countably infinite sets u, v.

In our method of procedure, and therefore in this formula, we may change u into U, and V into v, if at the same time we interchange h and k, and therefore write $\bar{k}$ in place of $\bar{h}$.

20. What we have first to consider is the circumstances under which we can proceed to the limit in the formula (V) under the sign of integration. We shall employ the known pair of sufficient conditions*

(1) *the integrands have a unique limit, when* $\bar{h} \to 0$, *except for a set of values of* v *of content zero; and*

(2) *the integrands form a bounded succession, or, more generally, are all numerically less than one and the same summable function.*

It is, of course, not necessary for the existence of the unique limit in (V), that the integrand should have a unique limit. In point of fact in §§ 24–31 we shall utilise this circumstance to obtain more general conditions. It would seem, however, that even the results we are about to obtain in the paragraphs which immediately follow, constitute a considerable advance on any hitherto formulated.

21. Making use of the conditions (1) and (2) just stated, we are able to prove the following theorem :—

THEOREM 2.—*If* $x(u, v)$, $y(u, v)$ *are functions satisfying the following conditions :—*

(i) *The four partial derivates* $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}$ *are bounded functions of* (u, v); *and*

---

* W. H. Young, "On Semi-integrals and Oscillating Successions of Functions," § 27. More general conditions are given in § 23. I take this opportunity of pointing out that, in § 24 of that paper, the reasoning employed does not prove the theorem stated, but only that obtained from it by adding the restriction that the succession should be bounded below (above). The following corrections are therefore necessary :—

P. 309, between lines 3 and 4, *read* "bounded below (above)," and line 5, *delete* "no."
Line 6, *delete* "positive (negative)," and *insert* "zero for unique double limit."
After line 6, *insert* "If the succession is bounded below, we may suppose the functions all positive, so that it is sufficient to show that there is no positive double limit."
Line 7, *after* "obvious" *insert* "then."
P. 310, § 25. In the enunciation *insert after* "functions" the words "bounded below (above)," and *delete* in the last line "above (below)."

The corresponding corrections in the proof and in § 26 can be easily supplied by the reader.

(ii) $\frac{\partial x}{\partial u}$ *and* $\frac{\partial y}{\partial u}$ *are continuous with respect to* v, *except at the points* (u, v) *of a plane set of content zero ; or*

$\frac{\partial x}{\partial v}$ *and* $\frac{\partial y}{\partial v}$ *are continuous with respect to* u, *except at the points* (u, v) *of a plane set of content zero.**

Suppose first for definiteness that $\frac{\partial y}{\partial v}$ and $\frac{\partial x}{\partial v}$ have the continuity properties specified in the statement of the theorem, so that, in particular, $\frac{\partial y}{\partial v}$ is for each fixed value of v, which does not belong to a certain set of values of v of zero content, continuous with respect to u, except for a set of values of u of zero content.

Then, except for a set of values of v of zero content,

$$\sum_{a}^{c} \{x(u+h, V)-x(u, V)\} \frac{\partial y(u, V)}{\partial V},$$

which is the first term in the integrand of V, certainly has a unique limit

---

* It is this theorem which corresponds most closely with that obtained by E. W. Hobson in these *Proceedings*, Ser. 2, Vol. 8 (1909), pp. 22–39, "On some Fundamental Properties of Lebesgue Integrals in a Two-Dimensional Domain." More especially pp. 32–39.

Though only a special case of the general theorem obtained below, it appears to differ essentially only from that due to Hobson in the omission of some of Hobson's restrictions. These latter are seen in fact to be unnecessary in the light of the methods employed in the present paper. Among these unnecessary restrictions, it may be particularly noted that in our statement the set of zero content need not be closed, moreover it is not necessary to hypothecate that the corresponding (x, y)-set should have zero content.

The most striking simplification is that which corresponds to the omission of all those conditions entailed by the supposition, necessarily made by Hobson in his method of treatment, that the correspondence between the two planes should be (1, 1), and that further it should be what he calls a "normal" correspondence, this latter involving quite a series of conditions given on p. 37 *loc. cit.*, relating to an intermediate transformation, obtained by eliminating v, between the equations giving x and y in terms of u and v.

It may fairly be contended that this series of conditions is only introduced *ad hoc*. They constitute a part of the scaffolding, and *a priori* reasons would accordingly lead to the conclusion that their removal must be possible. If I lay stress on this point it is because nothing could better illustrate the necessity of a treatment of area such as that employed in the present paper.

It may be added that, while in theory the limitations there introduced are serious, in practice the determination of whether the correspondence is "normal" or not, even if the elimination could be carried out, might well present difficulties.

when $h \to 0$. In fact, as $\frac{\partial x}{\partial u}$ is bounded, $x(U, V)$ is an integral with respect to U, and therefore $\frac{\partial y}{\partial V}$, regarded as a function of U, has an integral with respect to $x(U, V)$ for each fixed value of V not belonging to a set of zero content, namely,

$$\int_a^c \frac{\partial y(U, V)}{\partial V}\, dx(U, V) = \int_a^c \frac{\partial x}{\partial U}\,\frac{\partial y}{\partial V}\,\partial U.$$

Indeed the condition for the existence of this "Riemann" integral is that, V being fixed, the variation of $x(U, V)$ over the set of points U where $\frac{\partial y}{\partial V}$ is discontinuous with respect to u, should be zero: this will be the case provided this set of points u has zero content, since $x(u, v)$ is an integral with respect to u. Hence by the definition of a Riemann integral with respect to a function of bounded variation, the first term of the integrand in (V) has this "Riemann" integral for limit, when $\bar{h} \to 0$.

Precisely the same reasoning applies of course to the second term of the integrand in (V), interchanging x and y. Thus this second term tends to

$$\int_a^c \frac{\partial y}{\partial U}\,\frac{\partial x}{\partial V}\, dU,$$

as $h \to 0$.

On the other hand, the first case of condition (2) of § 20 is clearly satisfied, since the integrand in (V) consists only of bounded functions, $x(u, v)$ and $y(u, v)$ being now necessarily continuous functions of (u, v), owing to the boundedness of all their derivates.

If secondly it be $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ which have the continuity properties, the argument is the same, except that the parts played by u and v are to be interchanged, the corresponding alteration being also effected in the formula (V). In this case $\bar{h}$ tends first to zero and then $\bar{k}$.

Thus our theorem is proved.

22. It should be remarked that our reasoning does not permit us to deduce the same result from the possession of the continuity property by $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial v}$, for example. This will be clear if we reflect that, though A is known to be the unique double limit of our summation, it is not known

that the two constituent summations, whose difference is the whole summation, have each a unique double limit. Our method shews that, with the hypothesis of Theorem 2, they both have repeated limits in the particular order determined by whichever of the alternative hypotheses (ii) we take. The difference of these two repeated limits may be therefore used to calculate A as a repeated limit in the same order. But we should not be justified in asserting that A was the difference of the repeated limit of one constituent summation in one order and the repeated limit in the other order of the second summation.

23. It will be noticed that the reasoning employed in proving Theorem 2 really gives us more than is stated in that theorem. But it does not seem worth while to delay passing on to the general result of the paper. As, however, in Theorem 2, the derivates are all bounded, it is perhaps worth while to give the following simple enunciation, which avoids this assumption, though in some respects its demands are greater than those of Theorem 2, or even of Theorem 1.

THEOREM 3.—*If $x(u, v)$ and $y(u, v)$ are functions of bounded variation of (u, v), and are integrals with respect to u for each fixed value of v, and integrals with respect to v for each fixed value of u, then*

$$A = \int_a^c \int_b^d \frac{\partial(x. y)}{\partial(u, v)}\, du\, dv.$$

We may again employ the formula (V).

That the integrand in this formula has a unique limit follows from a fundamental property of a function of bounded variation in two variables,* namely, that the mixed differential coefficients of the second order necessarily exist and are finite, except for a plane set of content zero. Hence it follows that the continuity property in the enunciation of Theorem 2 holds for the partial derivates of the first order, and therefore, as before, each of the two terms in the integrand in (V) leads to a unique limit when $\overline{h} \to 0$.

Further, that we may introduce the limit under the integral sign, is easily seen as follows.

---

* W. H. Young, "Sur la dérivation des fonctions à variation bornée", *Comptes Rendus*, Vol. 164 (1917).

We may write

$$x(u, v) = x_1(u, v) - x_2(u, v),$$

$$y(u, v) = y_1(u, v) - y_2(u, v),$$

where each of the functions x_1, x_2, y_1 and y_2 is a $(+, +)$-monotone increasing function of (u, v), that is a function whose increment with respect to each variable is a positive ($\geqslant 0$) monotone increasing function of the other variable: the same is therefore true when we replace the increment by the upper or lower derivate, or differential coefficient.

Hence
$$0 \leqslant \frac{\partial y_i(u, v)}{\partial v} \leqslant \frac{\partial y_i(c, v)}{\partial v}, \quad (i = 1 \text{ or } 2),$$

and
$$\frac{\partial y_i(c, v)}{\partial v} = \mu_i(v),$$

being one of the derivates of a monotone function of v, is a summable function of v.

Therefore, for $i = 1$ or 2, $j = 1$ or 2,

$$0 \leqslant \sum_a^c \{x_j(u+h, v) - x_j(u, v)\} \frac{\partial y_i}{\partial v} \leqslant \mu_i(v) \sum_a^c \{x_j(u+h, v) - x_j(u, v)\}$$

$$\leqslant \mu_i(v) \{x_j(c, v) - x_j(a, v)\} \leqslant \mu_i(v) \{x_j(c, d) - x_j(a, b)\}.$$

Thus the first term in the integrand in (V) becomes the sum of four terms with proper signs, each of which is $\geqslant 0$ and not greater than a summable function of v. The same applies to the second term in the integrand of (V).

For each of these terms, therefore, the more general criterion (2) of § 20 is satisfied, while a unique limit when $\bar{h} \to 0$ exists, since x_i and y_i, like x and y, are functions of (u, v) of bounded variation. Thus we may, for each of these eight terms separately, introduce the limit under the integral sign, and therefore we may proceed to the limit under the integral sign, when the integrand is that given in (V). This proves the theorem.

24. Hitherto we have required that the integrand in (V) should, except at a set of content zero, itself have a unique limit when $\bar{h} \to 0$; in other words, we have based our reasoning on the theory of integrable successions of functions. The moment we let this hypothesis drop, we require the methods of the much more difficult theory of the integrability of oscillating successions of functions.

If we suppose the integrand in (V) to be bounded, we are at once sure that the limits of the two terms whose difference is the summation tending to A as limit, must lie between the integrals of the upper and lower limits of the corresponding part of the integrand, and accordingly between the integrals with respect to v of the "Darboux" upper and lower integrals of $\frac{\partial y}{\partial v}$ taken with respect to $x(u, v)$, for constant v, and of $\frac{\partial x}{\partial v}$ taken with respect to $y(u, v)$, for constant v. As the "Lebesgue" integral always lies between the "Darboux" upper and lower integrals, it is natural to attempt to formulate theorems in which the "Lebesgue" integral, though not approached by the integrand is, when integrated with respect to v, the required limit.

25. For our present purpose it is convenient to write the formula (V) as follows:—

$$A = \operatorname*{Lt}_{\bar{h}\to} \tfrac{1}{2} \int_b^d dV \sum_a^c \left[\int_u^{u+h} \left\{ \frac{\partial x(U, V)}{\partial U} \frac{\partial y(u, V)}{\partial V} - \frac{\partial y(U, V)}{\partial U} \frac{\partial x(u, V)}{\partial V} \right\} dU. \right. \tag{VI}$$

We shall now transform the two terms under the limit, writing, for instance, the first of these in the form

$$\int_b^d dV \int_a^c \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} dU, \tag{α}$$

where we have introduced an auxiliary function ϕ_r at the r-th stage in our passage to the limit, when $\bar{h}$ assumes the value $\bar{h}_r$, of a certain sequence

$$\bar{h}_1 > \bar{h}_2 > \ldots > \bar{h}_r > \ldots \to 0.$$

This function $\phi_r(U, V)$ is defined as equal to $y(u, V)$ in the half-open interval $(u \leqslant U < u+h)$, for each of these intervals in the division made at the r-th stage. *Accordingly $\phi_r(U, V)$ is, for constant V, constant in stretches.*

Taking any point U, such that

$$u < U < u+h,$$

we then have $$\phi_r(U, V) = y(u, V), \tag{β}$$

so that, *for constant* U, $\phi_r(U, V)$ *is an integral*, and

$$\frac{\partial \phi_r(U, V)}{\partial V} = \frac{\partial y(u, V)}{\partial V}, \qquad (\gamma)$$

which gives the expression (α) as equivalent to the first term under the limit sign in (VI).

26. As we proceed, stage by stage, the point U being fixed, the point u will either coincide with U, or will move up to u as limit. In either case, since $y(U, V)$ is a continuous function of U, $y(u, V)$ tends to $y(U, V)$ as unique limit, when $u \to U$. Thus

$$\operatorname*{Lt}_{r\to\infty} \phi_r(U, V) = y(U, V).$$

Since therefore $y(U, V)$ and $\phi_r(U, V)$ are both integrals with respect to V, we may write

$$\operatorname*{Lt}_{r\to\infty} \int_b^V \frac{\partial \phi_r(U, V)}{\partial V} dV = \int_b^V \frac{\partial y(U, V)}{\partial V} dV. \qquad (\delta)$$

27. We consider first, for simplicity, the case when the partial derivates of x and y with respect to u and v are all bounded. It will be seen that we require no further condition, if this be assumed. We have, in fact, the following theorem :—

THEOREM 4.—*If the partial derivates* $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ *are all bounded, then*

$$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)} du\, dv.$$

For, if the derivates all lie between $\pm M$, the functions $\frac{\partial \phi_r}{\partial V}$ also lie between $\pm M$, for all values of r, U and V, by reason of (γ) : they form therefore, as $r \to \infty$, a bounded succession, whose integrals converge to an integral, by the relation (δ). By a well-known theorem,* we may then insert the bounded function $\frac{\partial u(U, V)}{\partial U}$ under the integral sign on both sides of the relation (δ). Thus

$$\operatorname*{Lt}_{r\to\infty} \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} dV = \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial y(U, V)}{\partial V} dV.$$

* W. H. Young, "Successions of Integrals and Fourier Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11 (1912), p. 62, see also below § 32.

Since these integrands, and therefore the integrals, are bounded functions of U, we may now integrate term-by-term with respect to U, and get

$$\operatorname*{Lt}_{r\to\infty}\int_a^c dU\int_b^d \frac{\partial x}{\partial U}\frac{\partial \phi_r}{\partial V}dV=\int_a^c dU\int_b^d \frac{\partial x}{\partial U}\frac{\partial y}{\partial V}dV.$$

Changing the order of integration, we see by (α) that the first term under the limit in (VI) leads, when $r\to\infty$, that is $\bar{h}\to 0$, to the integral on the right in the last equation. Interchanging x and y, as we may, since the hypotheses are symmetrical, we get the second term in (VI). Thus, subtracting, we get the required result.

28. We have stated and proved Theorem 4 separately, because of its great simplicity of treatment, and the relative ease with which it is proved, oscillating successions being particularly easy to deal with when they are bounded. This facility, however, is really largely due to the fact that we are then able to assert with certainty that the integral $\int_E f_n(x)\,dx$ of the typical function $f_n(x)$ of the succession over a set E, has, as the content E tends to zero, and simultaneously $n\to\infty$, the unique double limit zero. Now the convergence of this integral to zero is also secured when $|f_n(x)|$, without being bounded, is less than or equal to a summable function, independent of n, and we are naturally led to enquire whether the condition of boundedness of the partial derivates may not be replaced by the corresponding more general condition. In the next theorem it will be proved that this is the case. We must, however, it will be noticed, specifically introduce into the conditions the fact that x and y are integrals with respect to each variable separately, a condition which fulfils itself when the derivates are bounded.

THEOREM 5.—*If*

(i) *$x(u, v)$ and $y(u, v)$ are integrals with respect to u, for each fixed value of v, and integrals with respect to v, for each fixed value of u;*

(ii) *each of the partial derivates of x and y with respect to v is, except at most for a set of values of v of content zero, numerically less than, or equal to, a summable function of v independent of u;**

* We might only postulate this condition for a countably infinite set of values of u, dense everywhere. We should, however, only formally gain in generality, since when the condition is satisfied in this form, it is satisfied in the form given in the text.

In fact, if for a countably infinite set of values of u, dense everywhere, we have, except

(iii) *the same as* (ii) *with u and v interchanged;*

then
$$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$

29. In the proof of Theorem 5 we shall make use of the following theorem on the integration of oscillating successions of functions, the proof will be given in a subsequent article:

If the integrals of a succession of summable functions $f_1(x), f_2(x), \ldots$ *converge to an integral,*
$$\operatorname{Lt}_{n\to\infty} \int f_n(x)\, dx = \int g(x)\, dx,$$

then we may insert under the integral signs any the same function $g(x)$, *provided* $\int_E f_n(x)\, dx$ *and* $\int_E f_n(x)\, g(x)\, dx$, *both have the unique double limit zero, when* $E \to 0$, $n \to \infty$, *and the various integrals exist.*

30. To prove Theorem 5, we start once more with the formula (VI),

---

for a set of values of V of content zero,
$$-\mu(V) \leqslant \frac{\partial}{\partial v} y(u, V) \leqslant \mu(V), \tag{1}$$
we see, integrating with respect to V, that
$$y(u, V) + \int \mu(V)\, dV \quad \text{and} \quad \int \mu(V)\, dV - y(u, V)$$
are monotone increasing functions of V. But, if we write
$$Z_1(U, V) = y(U, V) + \int \mu(V)\, dV,$$
$Z_1(U, V)$ will be, like $y(U, V)$, a continuous function of U, and therefore a monotone increasing function of V, since it is the limit of $Z_1(u, V)$, when u tends to U as limit by means of the points of our countable set. Similarly,
$$Z_2(U, V) = \int \mu(V)\, dV - y(U, V)$$
is a monotone increasing function of V. Therefore, except for a set of values of V of content zero,
$$-\mu(V) \leqslant \frac{\partial}{\partial v} y(U, V) \leqslant \mu(V), \tag{1a}$$
for *all* values of U. This proves the above statement.—[*Note added May 11th*, 1919.]

and consider the expression (α) of § 25, viz.

$$\int_b^d dV \int_a^c \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} dU. \qquad (\alpha)$$

Supposing $\lambda(U)$ and $\mu(V)$ to be the functions hypothecated greater than $\left| \frac{\partial x(U, V)}{\partial U} \right|$ and $\left| \frac{\partial y(U, V)}{\partial V} \right|$ respectively, we have, by (γ),

$$\left| \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} \right| = \left| \frac{\partial x(U, V)}{\partial U} \frac{\partial y(u, V)}{\partial V} \right| < \lambda(U)\mu(V).$$

Therefore, as $E \to 0$ and $r \to \infty$, we have, integrating over a set E,

$$\left| \int_E \frac{\partial \phi_r(U, V)}{\partial V} \partial V \right| \leqslant \int_E \mu(V) dV \to 0, \qquad (1)$$

and

$$\left| \int_E \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} \partial V \right| \leqslant \lambda(U) \int_E \mu(V) dV \to 0. \qquad (2)$$

Since then the integrals of the succession of functions $\frac{\partial \phi_r(U, V)}{\partial V}$, for constant U, converge to an integral, namely, as shewn in § 25,

$$\operatorname*{Lt}_{r \to \infty} \int \frac{\partial \phi_r(U, V)}{\partial V} \partial V = \int \frac{\partial y(U, V)}{\partial V} \partial V, \qquad (\delta)$$

it follows from the theorem on oscillating successions just quoted, that, for every fixed value of U,

$$\operatorname*{Lt}_{r \to \infty} \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} \partial V = \int_b^d \frac{\partial x(U, V)}{\partial U} \frac{\partial y(U, V)}{\partial V} \partial V,$$

since these integrals certainly exist.

Moreover, since each of these integrals is numerically

$$\leqslant \lambda(U) \int_b^d \mu(V) dV,$$

that is a summable function of U, we may integrate the equation last obtained term-by-term with respect to U, and write

$$\operatorname*{Lt}_{r \to \infty} \int_a^c dU \int_b^d \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} dV = \int_a^c dU \operatorname*{Lt}_{r \to \infty} \int_b^d \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} dV = \int_a^c dU \int_b^d \frac{\partial x}{\partial U} \frac{\partial y}{\partial V} dV,$$

whence, the hypotheses being symmetrical with respect to x and y, the required result follows, as before (§ 27), from (VI).

31. *We may here remark that the assumptions made in Theorem* 5 *secure the boundedness with respect to* (u, v) *of the variation of each of the dependent variables* x *and* y *with respect to one of the independent variables* u, v, *the other being regarded as a parameter.* Hence also x and y are themselves bounded functions of (u, v).

Indeed, by (i), the total variation of x is $\int \left|\frac{\partial x}{\partial u}\right| du$, v being regarded as a parameter, and

$$|x(u,v)-x(a,v)| = \left|\int_a^x \frac{\partial x}{\partial u}\,du\right| \leqslant \int_a^x \left|\frac{\partial x}{\partial u}\right| du \leqslant \int_a^c \lambda(u)\,du,$$

where $\lambda(u)$ is the summable function hypothecated as $\geqslant \left|\frac{\partial x}{\partial u}\right|$. This suggests that this property, which does not seem to entail necessarily the hypotheses of Theorem V, may be used in place of one of the conditions there imposed. It is easily seen, in fact, that our reasoning in § 30 was more general than is required for the mere proof of Theorem V. We have accordingly the following slightly more general, if less elegant theorem :—

THEOREM VI.—*If*

(i) $x(u, v)$ *and* $y(u, v)$ *are integrals with respect to* u *for each fixed value of* v, *and integrals with respect to* v *for each fixed value of* u ;

(ii) $$\left|\frac{\partial y(u, V)}{\partial V}\right| \leqslant \mu(V), \quad \left|\frac{\partial x(u, V)}{\partial V}\right| \leqslant M(V),$$

where $\mu(V)$ *and* $M(V)$ *are summable functions of* V *independent of* u ;*

(iii) *the total variations of* $x(U, V)$ *and* $y(U, V)$ *with respect to* U *for constant* V *are bounded functions of* V *and therefore of* (U, V) ; *or more generally possess absolutely convergent integrals with respect to the integrals of* $\mu(V)$ *and* $M(V)$ *respectively* ;

then $$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$$

For, if $X(U, V)$ denote the total variation of $x(U, V)$ for constant V,

* This condition need only hold for a countable everywhere dense set of values of u.

we have, by (i),

$$X(U, V) = \int \left| \frac{\partial x(U, V)}{\partial U} \right| dU,$$

and, by (iii), is either a bounded function of V, or is at any rate such that $\mu(V)\, X(U, V)$ is summable with respect to V,

$$\int \mu(V)\, X(U, V) dV = \int dV \int \mu(V) \left| \frac{\partial x(U, V)}{\partial U} \right| dU.$$

Therefore, since the integrand is positive, we may change the order of integration, and write

$$\int dU \int \mu(V) \left| \frac{\partial x(U, V)}{\partial U} \right| dV = \int dV \int \mu(V) \left| \frac{\partial x(U, V)}{\partial U} \right| dU.$$

Hence

$$u(V) \left| \frac{\partial x(U, V)}{\partial U} \right|$$

is a summable function of V, except possibly for a set of values of U of content zero. Therefore, excepting these values of U,

$$\left| \int_E \frac{\partial x(U, V)}{\partial U} \frac{\partial \phi_r(U, V)}{\partial V} dV \right| \leqslant \int_E \left| \frac{\partial x(U, V)}{\partial U} \right| \mu(V) dV \to 0,$$

as in (2) in the proof of Theorem 5. The relation (1) of course still holds, as well as the relation (δ) of § 26. Therefore except for a set of values of U of content zero,

$$\operatorname*{Lt}_{r \to \infty} \int_b^d \frac{\partial x}{\partial U} \frac{\partial \phi_r}{\partial V} dV = \int_b^d \frac{\partial x}{\partial U} \frac{\partial y}{\partial V} \partial V.$$

But these integrals, which certainly exist, are numerically

$$\leqslant \int_b^d \left| \frac{\partial x}{\partial U} \right| \mu(V)\, dV,$$

which is a summable function of U, its integral being, as we saw,

$$= \int_b^d \mu(V)\, X(U, V)\, dV.$$

Therefore we may integrate our equation term-by-term with respect to U, whence as before the theorem follows.

PART III.—*On Oscillating Successions of Functions.*

32. We now proceed to prove the theorem on successions, quoted in § 29, for any number of variables.

If x denote any ensemble of variables, finite in number [*e.g.* $x = (u, v)$, when we are working in a plane], and the integral sign be understood to imply multiple integration of the proper number of dimensions, a summable function $f(x)$ has the characteristic property that its integral over any set of points of content E tends to zero, as $E \to 0$. We write this

$$\int_E f(x)\, dx \to 0,$$

reserving the symbol $\int f(x)dx$ for indefinite integration over any "interval" (rectangle, block, &c., according to the number of dimensions of x). In particular, if $f(x)$ is independent of one or more of the variables implied in x, this property holds; or if $f(x) = f_1(u)\, f_2(v)$, is the product of a function of certain of the variables, denoted by u, and a function of the remaining variables, denoted by v.

We have already used [tacitly in (2) of § 19] this property of a summable function of a single variable, in connection with the theorem on successions of functions which states that the analogous condition

$$\operatorname*{Lt}_{\substack{n\to\infty \\ E\to 0}} \int_E f_n(x)\, dx = 0,$$

where the limit is now a double one, is sufficient in order that the succession of summable functions $f_1(x)$, $f_2(x)$, ..., which converges except at a set of content zero to a summable function $f(x)$, should be *absolutely integrable*, that is,

$$\operatorname*{Lt}_{n\to\infty} \int_S f_n(x)\, dx = \int_S f(x)\, dx,$$

integration being over any and every set of points S.

As already remarked, this same condition plays a part of some importance, when we merely know that the *integrals* of the succession converge to an integral. We have, in fact, the following general theorem:—

THEOREM.—*If the integrals of the succession of summable functions*

$$f_1(x),\ f_2(x),\ \ldots,\ f_n(x),\ \ldots$$

converge to an integral

$$\operatorname*{Lt}_{n\to\infty} \int f_n(x)\,dx = \int g(x)\,dx, \tag{1}$$

and the usual condition is satisfied, namely,

$$\operatorname*{Lt}_{\substack{n\to\infty \\ E\to 0}} \int_E f_n(x)\,dx = 0, \tag{2}$$

then we may insert under the integral sign in (i) *any the same function* $g(x)$, *such that all the functions* $f_n(x)\,g(x)$, *as well as* $q(x)\,g(x)$, *are summable, and*

$$\operatorname*{Lt}_{\substack{n\to\infty \\ E\to 0}} \int_E f_n(x)\,g(x)\,dx = 0.$$

Moreover in this case integration may be over any set S.

COR. 1.—*The theorem holds if for all values of n,*

$$|f_n(x)| \leqslant \phi(x),$$

where $\phi(x)$ *is a summable function of the variables x, such that* $\phi(x)\,g(x)$ *is summable.*

COR. 2.—*If x denote the ensemble of certain variables, denoted by u, and certain other variables, denoted by v, it is sufficient in order that we may insert a function* $g(x)$ *under both integral signs in the relation* (*supposed to hold*)

$$\int f_n(x)\,dx \to \int q(x)\,dx, \tag{1}$$

if for all values of n,

$$|f_n(x)| < \phi(u), \qquad |q(x)| < \psi(v),$$

where $\phi(u)$ *and* $\psi(v)$ *are functions only of the variables implied in u and v respectively.*

33. The proof of these results in the explicit case of two variables $x = (u, v)$, is given here for reference; the reasoning is perfectly general for any number of variables, when we regard u and v as implying each a combination of other variables.

The proof of the main theorem requires certain Lemmas.

LEMMA 1.—*If*
$$\operatorname*{Lt}_{\substack{n\to\infty\\E\to 0}} \int_E f_n(x)\,dx = 0,$$

then also
$$\operatorname*{Lt}_{\substack{n\to\infty\\E\to 0}} \int_E |f_n(x)|\,dx = 0.$$

For let E be any set and divide it into two parts E'_n and E''_n, such that in E'_n the function $f_n(u, v)$ is $\geqslant 0$, and in E''_n it is < 0. Then, if $E \to 0$, $E'_n \to 0$, and $E''_n \to 0$. Therefore the first of the above relations is satisfied when we replace E by E'_n, or by E''_n; subtracting we therefore get the second of the above relations.

LEMMA 2.—*If, integrating over any rectangle,*

$$\int f_n(x)\,dx \to \int q(x)\,dx, \quad (n \to \infty), \tag{1}$$

then the same is true over any set of points, provided

$$\operatorname*{Lt}_{\substack{n\to\infty\\E\to 0}} \int_E f_n(x)\,dx = 0. \tag{2}$$

Indeed (1) being true over any rectangle, is true over any finite number of non-overlapping rectangles. Now, if S be any closed plane set, we can enclose it in a finite number of squares, having points of S as centres, and sides of length e parallel to any two given directions at right angles. These squares in general overlap; but producing their sides indefinitely, we divide them up into non-overlapping rectangles, finite in number, containing the whole set S inside them or on their perimeters.

Denoting these rectangles by D and the complementary set to S in these rectangles by E, we see that, as $e \to 0$, the content E also tends to zero. For otherwise there would be at least one point belonging to all the sets E, and this is not the case, since S is a closed set, and is therefore itself the complete inner limiting set of the set of squares, as $e \to 0$.

Thus
$$\int_D f_n(x)\,dx = \int_S f_n(x)\,dx + \int_E f_n(x)\,dx,$$

where the latter integral tends, by our hypothesis, to zero when $e \to 0$, and therefore, as we have seen $E \to 0$.

Again,
$$\int_D q(x)\,dx = \int_S q(x)\,dx + \int_E q(x)\,dx,$$

where the latter integral also tends to zero, by the fundamental property of a summable function of $x = (u, v)$.

Thus, since (1) has been shewn to hold for integration over D, so that

$$\int_S f_n(x)\,dx + \int_E f_n(x)\,dx \to \int_S q(x)\,dx + \int_E q(x)\,dx,$$

we have in consequence of what has been pointed out

$$\int_S f_n(x)\,dx \to \int_S q(x)\,dx.$$

This proves that (1) holds for integration over any closed set.

But if S is any measurable plane set, it is the sum of a closed plane closed set Σ and a set E whose content may be taken to be as small as we please. Thus we express our double integral over S as the sum of one over Σ and another over E, of which the last tends to zero, if we take a succession of such sets (Σ, E) with $E \to 0$. Since the equation (1) holds for integration over the closed set Σ, we have

$$\int_S f_n(x)\,dx - \int_E f_n(x)\,dx \to \int_S q(x)\,dx - \int_E q(x)\,dx,$$

which leads again to

$$\int_S f_n(x)\,dx \to \int_S q(x)\,dx,$$

since the other two integrals tend to zero when $E \to 0$. This proves the equation (1) for integration over any measurable set S.

LEMMA 3.—*If in addition to the conditions* (1) *and* (2) *of Lemma* 2, *we have*

$$|f_n(u, v)| \leqslant \phi(u), \tag{3}$$

we may take $q(u, v)$ *in* (1) *to be such that*

$$|q(u, v)| \leqslant \phi(u). \tag{4}$$

For, if not, for some point (u, v),

$$\phi(u) < |q(u, v)|,$$

and therefore

$$\phi(u) + e < |q(u, v)|,$$

for some value of e.

Let us take together all the points (u, v) at which this last relation

holds good for one and the same e, and $q(u) \geqslant 0$, and denote this set by S. We then have, by Lemma 2, for a sufficiently large n,

$$\iint_S [\phi(u)+e]\,du\,dv \leqslant \iint_S q(u,v)\,du\,dv \leqslant \iint_S f_n(u,v)\,du\,dv+\tfrac{1}{2}eS,$$

which is obviously only possible if S is a set of zero content. Since this is true for all positive values of e, it follows that, except at a set of content zero, we have, when $q(u, v)$ is not negative,

$$q(u,v) \leqslant \phi(u).$$

Similarly, except at a set of content zero, when $q(u, v)$ is negative,

$$-\phi(u) \leqslant q(u,v).$$

LEMMA 4.—*If* $$\iint_E f_n(u,v)\,du\,dv \to 0,$$

the succession of integrals

$$\iint_S f_n(u,v)\,du\,dv$$

is bounded and oscillates uniformly and homogeneously.

To prove that the succession of integrals is bounded, let us determine E_0 and u_0, so that, for $E < E_0$ and $u > u_0$,

$$-e < \iint_E f_n(u,v)\,du\,dv < e.$$

Next let us determine an integer k so that

$$kE_0 > (d-b)(c-a).$$

Then, if S be any set in the rectangle $(a, b\,;\, c, d)$, and we divide the whole rectangle into k equal parts, each of these parts, and therefore each of the sub-sets of S in them, has content less than E_0. Hence, by the above, for $n \geqslant n_0$,

$$-ke < \iint_S f_n(u,v)\,du\,dv < ke.$$

Thus the whole set of integrals, from $u = u_0$ onwards, is bounded, and therefore the whole set is bounded.

To prove the oscillation uniform and homogeneous, let us add to and subtract from the set S any sets E_1, E_2, where $E_1 \to 0$, $E_2 \to 0$. Then, denoting the new set by T,

$$\iint_T f_n(u, v)\,du\,dv = \iint_S f_n(u,v)\,du\,dv + \iint_{E_1} f_n(u, v)\,du\,dv - \iint_{E_2} f_n(u,v)du\,dv.$$

By hypothesis the second and third integrals on the right tend to zero. Therefore the lowest and highest double limits on the left are respectively the lowest and highest limits of $\iint_S f_n(u, v)\,du\,dv$. In other words, the succession of integrals oscillates uniformly. Moreover, this is true whether we let u describe all integers or only a sub-set of integers tending to infinity; thus the oscillation is uniform and homogeneous.

This proves the theorem.

COR. 1.—*The results of the theorem are still true if we substitute* $|f_n(x)|$ *for* $f_n(x)$.

COR. 2.—*There is in every succession of the integrals a subsuccession which converges uniformly to an integral.*

COR. 3.—*If the integrals converge, they converge uniformly to an integral.*

34. We can now prove the theorem given in § 32.

THEOREM.—*If*
$$\int_a^c \int_b^d f_n(u, v)\,du\,dv \to \int_a^c \int_b^d q(u, v)\,du\,dv, \tag{1}$$

and, when $E \to 0$, $n \to \infty$,

$$\iint_E f_n(u, v)\,du\,dv \to 0, \tag{2}$$

then we may insert in (1) *under each integral sign any function* $g(u, v)$ *such that* $f_n(u, v)\,g(u, v)$ *for all values of* n *and* $g(u, v)\,q(u, v)$ *are summable, and when* $E \to 0$, $n \to \infty$,

$$\iint_E f_n(u, v)\,q(u, v)\,du\,dv \to 0. \tag{3}$$

For, by (2), using Lemma 2,

$$\iint_S f_n(u, v)\, du\, dv \to \iint_S q(u, v)\, du\, dv, \tag{4}$$

where S is any set.

This remains true when we insert any constant c under the integral sign on both sides, and therefore, also, when we add together any finite number of such relations. That is, *denoting by $g(u, v)$ a function which assumes only a finite number of values :*

$$\iint_S f_n(u, v)\, g(u, v)\, du\, dv \to \iint_S q(u, v)\, g(u, v)\, du\, dv. \tag{5}$$

But any bounded function $g(u, v)$ being given, we can find a function of the type just considered, say $g_r(u, v)$ differing from $g(u, v)$ at every point by less than $1/r$. Hence, as $n \to 0$,

$$\int_a^c \int_b^d f_n(u, v)\, g(u, v)\, du\, dv - \int_a^c \int_b^d f_n(u, v) \{g(u, v) - g_r(u, v)\}\, du\, dv$$

$$\to \int_a^c \int_b^d q(u, v) g(u, v)\, du\, dv - \int_a^c \int_b^d q(u, v) \{(u, v)\, g_r(u, v)\}\, du\, dv,$$

or, which is the same thing,

$$\int_a^c \int_b^d f_n(u, v) g(u, v)\, du\, dv - \theta \int_a^c \int_a^d |f_n(u, v)|\, du\, dv/r$$

$$\to \int_a^c \int_b^d q(u, v) g(u, v) du\, dv - \theta' \int_a^c \int_b^d |q(u, v)|\, du\, dv/r,$$

where $|\theta| \leqslant 1$, $|\theta'| \leqslant 1$.

But, by Lemma 4, the second terms on each side are numerically less than a finite constant divided by r, and therefore are as small as we please, r being large enough. Thus we obtain again the relation (3), $g(u, v)$ being any bounded function.

Finally, when $g(u, v)$ is merely summable, we divide it into the difference of two functions, each of which is $\geqslant 0$ and summable. It is therefore only necessary to consider one of these.

Let then $g(u, v) \geqslant 0$, and define an auxiliary bounded function $g_r(u, v)$ equal to $g(u, v)$ except at the points of the set E where $g(u, v) > r$, since $g(u, v)$ is summable, $E \to 0$ when $r \to \infty$.

Since, by what has been shewn, the theorem is true for $g_r(u, v)$, we have, since $g - g_r = 0$, except on E when $g - g_r = g$,

$$\int_a^c \int_b^d f_n(u, v)\, g(u, v)\, du\, dv - \iint_E f_n(u, v) g(u, v)\, du\, dv$$

$$\to \int_a^c \int_b^d q(u, v)\, g(u, v)\, du\, dv - \iint_E q(u, v) g(u, v)\, du\, dv.$$

By (3), the relation (5) again results, since by hypothesis $g(u, v)\, q(u, v)$ is summable. This proves the relation (5) for any summable function $g(u, v)$ satisfying the conditions of the enunciation. Hence the theorem is true. The proof of the corollaries may be left to the reader.

THE SIGNIFICANCE OF APOLAR TRIANGLES IN ELLIPTIC FUNCTION THEORY

By Prof. WILLIAM P. MILNE *and* Dr. D. G. TAYLOR.

[Received May 23rd, 1919.—Read June 12th, 1919.]

Introduction.

1. The theory of apolarity in its bearing on the geometry of rational curves has received much attention for a considerable time. A good account of the chief results obtained and the general trend of investigations is to be found in the treatise *Apolarität und Rationale Curven*, by Dr. W. Franz Meyer, at the end of which a substantial list of references to original memoirs is appended. Comparatively little, however, is as yet known of the applications of apolarity to curves which are not rational, but Caporali devotes a paper "Teoremi sulle curve del terzo ordine," to the discussion of "conjugate triangles" (with respect to cubic curves), in his published *Memorie di Geometria*, p. 47. Gonseth also makes use of the apolar locus of two class-cubics, and in particular of class-cubics consisting of three points, in three papers in the *Nouvelles Annales*, 1917, pp. 125, 297, 421.

Apolar triangles are such that the bipolar (or mixed polar) line of any two vertices with respect to a given cubic curve passes through the third vertex. It has been shewn in the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 237 and 238, that if the apolar triangle be inscribed in the given cubic, and if its vertices be projected through any point of the curve on to the curve again so as to form a new triangle, this latter triangle will also be of the apolar type. The attempt to explain the significance of this result in the theory of elliptic functions has led to a generalisation of the property as follows :—

If ABC be a triangle inscribed in a cubic curve, one and only one member of the syzygetic pencil of cubic curves (i.e. the pencil of cubic curves through the points of inflexion of the given cubic) is apolar to ABC. If A, B, C be projected through any point T of the cubic curve on to the curve again, so that the points A', B', C' are obtained, the triangle $A'B'C'$ remains apolar to the same member of the syzygetic pencil.

Let us now consider the matter from the point of view of elliptic functions. Any straight line is cut by any two members of the syzygetic pencil in two triads of points apolar to one another. Hence, if we take two points of one triad, their bipolar point with respect to the other triad must be the third point of the first triad. Consequently, the bipolar lines of any two points A, B on a cubic curve with respect to the members of the syzygetic pencil must pass through the third intersection X of the line AB with the cubic curve. We thus see that the bipolar line of two given points A, B on the cubic curve with respect to any member of the syzygetic pencil must cut the original cubic in three points, C_1, C_2 and the point X in which the line AB meets the curve. Although the triangles ABC_1 and ABC_2 possess the projective property above described, the triad ABX does not. This is explained by the fact that if we write down the condition for apolarity with respect to a member of the syzygetic pencil, between the points whose elliptic parameters are α, β, γ, and then fix α and β and regard the condition as an equation in γ, one of the solutions will be $-(\alpha+\beta)$. Furthermore, the condition for apolarity can be resolved into two factors

$$\Delta \,.\, A = 0.$$

We shall call Δ the "collinearity factor" and A the "apolarity factor." It will be found that A is a function of the differences of the parameters $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, and is therefore unaltered if for α, β, γ we substitute $-(\alpha+\tau)$, $-(\beta+\tau)$, $-(\gamma+\tau)$. On the other hand, Δ is not independent of τ, if we make the above substitutions. The projective property above described in the case of ABC_1 and ABC_2 but not of ABX is thus analytically explained.

In the present paper the proofs by elliptic functions will first of all be given and thereafter the same results will be obtained by the methods of synthetic geometry. The latter method will shew that the properties under investigation extend considerably the significance of the one-to-one correspondence used by Goursat in the *Nouvelles Annales* (1896, p. 20) to prove Salmon's theorem relative to the invariance of the cross-ratio of the four tangents from a point on the cubic to the curve itself, and later on used by other writers to establish the focal properties of circular cubics.

Investigation by Elliptic Functions.

2. We proceed, therefore, to express the condition that the three points whose parameters are α, β, γ on the cubic curve $x = \wp(u)$, $y = \wp'(u)$ shall be apolar to an assigned member of the syzygetic pencil,

and thereafter to factorize the expression thus obtained. It will, however, shed a good deal of preliminary light on the nature of the results to be expected if we consider first of all the very simple case of the cuspidal cubic $x = u^{-2}$, $y = -2u^{-3}$. The condition that the points α, β, γ be apolar to the syzygetic member $y^2x+\lambda(4x^3-y^2) = 0$ is, after a little reduction,

$$(\alpha+\beta+\gamma)\{2-\lambda[(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2]\} = 0.$$

The "collinearity factor" is therefore $(\alpha+\beta+\gamma)$, which does not remain independent of τ when for α, β, γ we substitute $-(\alpha+\tau)$, $-(\beta+\tau)$, $-(\gamma+\tau)$. The "apolarity factor" is

$$\{2-\lambda[(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2]\},$$

which, on the other hand, does remain independent of τ on making the above substitution. We see therefore that in the case of the cuspidal cubic there are two points C_1 and C_2 (other than X) which along with two assigned points A, B form apolar triangles with respect to a member of the syzygetic pencil (X being the third point of intersection of the line AB with the curve), and we see also that if A, B, C_1, C_2 be projected through any arbitrary point of the cubic on to the cubic again so that A', B', C_1', C_2' are obtained, then $A'B'C_1'$ and $A'B'C_2'$ are also apolar triangles with respect to the same member of the syzygetic pencil, whereas $A'B'X'$ (where X' is the projection of X according to the above definition) does not in general satisfy the condition for apolarity with respect to the given member of the syzygetic system.

We shall now deal with the general case. Consider the canonical cubic

$$S \equiv -y^2+4x^3-g_2x-g_3 = 0, \tag{1}$$

and its Hessian,

$$H \equiv y^2x-g_2x^2-3g_3x-\tfrac{1}{12}g_2^2 = 0. \tag{2}$$

If α, β, γ are the elliptic parameters of any three points on S, the condition that these points constitute a triad apolar to $\lambda S+H = 0$ is

$$\lambda S_{\alpha\beta\gamma}+H_{\alpha\beta\gamma} = 0, \tag{3}$$

where
$$S_{\alpha\beta\gamma} \equiv 12\wp(\alpha)\wp(\beta)\wp(\gamma)-\Sigma\wp'(\beta)\wp'(\gamma)-g_2\Sigma\wp(\alpha)-3g_3, \tag{4}$$

$$H_{\alpha\beta\gamma} \equiv \Sigma\wp(\alpha)\wp'(\beta)\wp'(\gamma)-g_2\Sigma\wp(\beta)\wp(\gamma)-3g_3\Sigma\wp(\alpha)-\tfrac{1}{4}g_2^2, \tag{5}$$

the summations referring to α, β, γ.

Consider next the expression $\lambda S_{\alpha\beta\gamma}(\tau)+H_{\alpha\beta\gamma}(\tau)$, where

$$\left.\begin{aligned} S_{\alpha\beta\gamma}(\tau) &\equiv 12\wp(\alpha+\tau)\wp(\beta+\tau)\wp(\gamma+\tau)-\Sigma\wp'(\beta+\tau)\wp'(\gamma+\tau) \\ &\qquad -g_2\Sigma\wp(\alpha+\tau)-3g_3 \\ H_{\alpha\beta\gamma}(\tau) &\equiv \Sigma\wp(\alpha+\tau)\wp'(\beta+\tau)\wp'(\gamma+\tau)-g_2\Sigma\wp(\beta+\tau)\wp(\gamma+\tau) \\ &\qquad -3g_3\Sigma\wp(\alpha+\tau)-\tfrac{1}{4}g_2^2 \end{aligned}\right\}. \tag{6}$$

Consider also the function

$$\Delta(\tau) \equiv \begin{vmatrix} \wp(\alpha+\tau) & \wp'(\alpha+\tau) & 1 \\ \wp(\beta+\tau) & \wp'(\beta+\tau) & 1 \\ \wp(\gamma+\tau) & \wp'(\gamma+\tau) & 1 \end{vmatrix}. \tag{7}$$

It is easy to prove in the usual way that

$$\lambda S_{\alpha\beta\gamma}(\tau)+H_{\alpha\beta\gamma}(\tau) \equiv -(\lambda m+b)\,\Delta(\tau), \tag{8}$$

where
$$m = \frac{\wp_2'-\wp_3'}{\wp_2-\wp_3} = \frac{\wp_3'-\wp_1'}{\wp_3-\wp_1} = \frac{\wp_1'-\wp_2'}{\wp_1-\wp_2} = -2(\xi_1+\xi_2+\xi_3),$$

$$b = \frac{\wp_2\wp_3'-\wp_2'\wp_3}{\wp_2-\wp_3} = \frac{\wp_3\wp_1'-\wp_3'\wp_1}{\wp_3-\wp_1} = \frac{\wp_1\wp_2'-\wp_1'\wp_2}{\wp_1-\wp_2}$$

$$= \tfrac{1}{3}\{(\wp_1'+\wp_2'+\wp_3')+2(\xi_1+\xi_2+\xi_3)^3\}, \tag{10}$$

where $\wp_1 \equiv \wp(\beta-\gamma)$, &c., and $y = mx+b$ is the equation to the line joining the points on the cubic, whose elliptic parameters are $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$.

We thus arrive at the fact that the general condition for apolarity $\lambda S_{\alpha\beta\gamma}+H_{\alpha\beta\gamma}=0$ resolves itself into two factors

(1) the collinearity factor $\Delta = 0$,

(2) the apolarity factor $\lambda m+b = 0$.

We have seen above that $\Delta = 0$ is merely the condition that the points α, β, γ are collinear and does not remain independent of τ, when for α, β, γ we substitute $-(\alpha+\tau)$, $-(\beta+\tau)$, $-(\gamma+\tau)$. On the other hand, the apolarity factor $\lambda m+b$ is a function of $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, and therefore remains independent of τ when the above substitution is made. We thus have the following general theorem:—

If three points, which are not collinear, lie on a cubic curve and are apolar to a given member of the syzygetic pencil of cubics, the three points

obtained by projecting the three given points through a point of the cubic curve on to the curve again are still apolar to the same member of the syzygetic pencil.

We can also find very simply a geometrical meaning for the vanishing of the apolarity factor $(\lambda m+b)$. Let us take as before the curve in its canonical form $x=\wp(u)$, $y=\wp'(u)$, and let the images of the three given points $A\equiv\alpha$, $B\equiv\beta$, $C\equiv\gamma$ in the x-axis be respectively $A_0\equiv-\alpha$, $B_0\equiv-\beta$, $C_0\equiv-\gamma$. We see then that the three collinear points $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ are where the curve is met again by the lines B_0C, C_0A, A_0B. Now, if $y=mx+b$ be the equation of the line on which lie $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, it is plain that the condition $\lambda m+b=0$ reduces this line to the form $y=m(x-\lambda)$, which as m varies represents a pencil of lines passing through the point in which the tangent to $\lambda S+H=0$ at the infinite point of inflexion cuts the x-axis. Hence

If A, B, C be three points on the curve $x=\wp(u)$, $y=\wp'(u)$ and A_0, B_0, C_0 their images in the axis of x, the three points in which the lines B_0C, C_0A, A_0B meet the curve again lie on a straight line which cuts the axis of x in the same point as the tangent at the infinite point of inflexion to that member of the syzygetic pencil of cubics, to which the triangle ABC is apolar.

The following particular cases are at once evident and are of great interest:

The condition that three points $A\equiv\alpha$, $B\equiv\beta$, $C\equiv\gamma$ shall be apolar to the cubic curve itself is $m=0$, i.e.

$$\xi(\beta-\gamma)+\xi(\gamma-\alpha)+\xi(\alpha-\beta)=0.$$

The condition that the three points $A\equiv\alpha$, $B\equiv\beta$, $C\equiv\gamma$ shall be apolar to the Hessian is $b=0$, i.e.

$$\{\wp'(\beta-\gamma)+\wp'(\gamma-\alpha)+\wp'(\alpha-\beta)\}+2\{\xi(\beta-\gamma)+\xi(\gamma-\alpha)+\xi(\alpha-\beta)\}^3=0.$$

If A, B, C be three points on the curve $x=\wp(u)$, $y=\wp'(u)$, apolar to the curve itself, and A_0, B_0, C_0 their images in the axis of x, then the three points in which the lines B_0C, C_0A, A_0B meet the curve again, lie on a straight line parallel to the axis of x.

If A, B, C be three points on the curve $x=\wp(u)$, $y=\wp'(u)$ apolar to the Hessian, and A_0, B_0, C_0 their images in the axis of x, then the three points in which the lines B_0C, C_0A, A_0B meet the curve again, lie on a straight line passing through the origin.

Proofs by Synthetic Geometry.

3. We shall now give synthetic proofs of the above results which apply only to the case of an inscribed triangle, apolar to the cubic itself. In § 4 we shall deal with the general case.

A triangle inscribed in a cubic curve and apolar to it, projects through a point of the curve on to the curve again into a triangle apolar to the given cubic.

Let A, B, C be a triangle on the cubic curve and apolar to it. Let T be any point on the curve and let AT, BT, CT intersect the curve again in A', B', C' respectively.

Consider the pencil of cubics passing through the intersections of the given cubic with the three lines ATA', BTB', CTC', and, in particular, consider the nodal members of the pencil. We know that any three points lying on three concurrent lines, one on each line, are apolar to the cubic consisting of the three lines. Hence ABC being apolar to the given cubic and also to ATA', BTB', CTC' must be apolar to every member of the pencil. But the necessary and sufficient condition that the triangle ABC be apolar to a nodal cubic on which it lies is that the Hessian line-pair with respect to the pencil of three lines which A, B, C subtend at the node N, shall intersect the nodal cubic again in two points, the tangents at which meet in a point on the curve. Now, if PTP' be three collinear points on the nodal cubic, it is plain that a one-one correspondence exists between NP and NP'. Hence the rays $N[A, B, C]$ correspond to the rays $N[A', B', C']$. Also, if $N[H, K]$ be the Hessian line-pair of $N[A, B, C]$, it follows that $N[H', K']$ is the Hessian line-pair of $N[A', B', C']$. But, it is a well-known property of cubic curves that two points whose tangents meet in a point of the curve project through a point of the curve into two points whose tangents also intersect in a point of the curve. Hence the tangents at H', K' intersect in a point of the curve, and the triangle A', B', C' must therefore be apolar to the nodal cubic. Finally, since $A'B'C'$ is apolar to *all* the nodal cubics of the system, it must be apolar to the original cubic. The theorem is thus established.

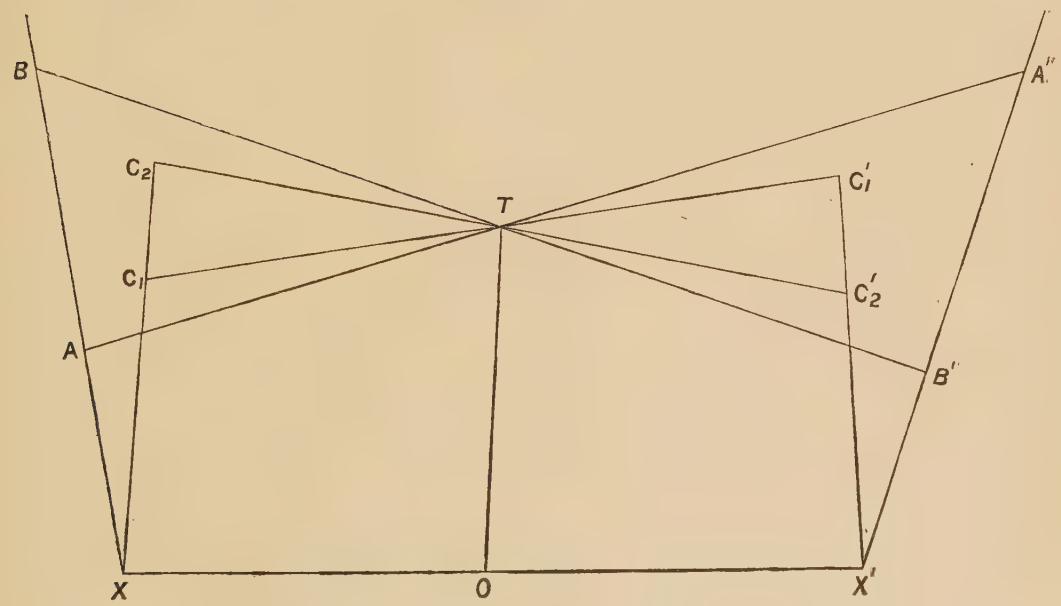
If ABC be a triangle inscribed in the cubic curve $x = \wp(u)$, $y = \wp'(u)$ and apolar to it, and if A_0, B_0, C_0 be the images of A, B, C in the axis of x, to prove that the three points, in which the lines B_0C, C_0A, A_0B cut the cubic again, lie on a straight line parallel to the axis of x.

It is plain that A, B, C and their images A_0, B_0, C_0 lie on a conic. Hence since the three straight lines B_0CF, C_0AG, A_0BH cut the cubic in

nine points of which the *six* A, B, C, A_0, B_0, C_0 lie on a conic, it is at once evident that F, G, H are collinear.

Let us now denote the point of inflexion at infinity by I and consider the pencil of cubics through the points of intersection of the given cubic with the lines IAA_0, IBB_0, ICC_0. If a member of the pencil cuts B_0C in F and C_0A in G, it is obvious that a one-to-one correspondence exists between F and G, and that the point of intersection of B_0C and C_0A is self-corresponding, whence FG always passes through a fixed point U. To find the position of this fixed point U, we consider *two* particular members of the pencil. One member consists of the conic $ABCA_0B_0C_0$ and the inflexional tangent at I, *i.e.* the line at infinity. Hence U lies on the line at infinity and all the chords FG are parallel. Next consider the three lines IAA_0, IBB_0, ICC_0, and let B_0C, C_0A, A_0B meet these lines in F, G, H respectively. It is easy to shew from elementary conic geometry that FGH is parallel to the axis of x, if ABC be apolar to the cubic consisting of the conic $ABCA_0B_0C_0$ and the inflexional tangent at I (*i.e.* the line at infinity), in other words if the point of which $ABCA_0B_0C_0$ is the polar conic with respect to the triangle ABC lie on the line at infinity. Hence the point U must be the point of intersection of the line at infinity and the axis of x, *i.e.* FGH is parallel to the axis of x.

The General Case.



4. *Let ABC be three points on a given cubic curve apolar to the curve*

S of the syzygetic pencil: to prove that if A, B, C be projected through any point T of the given cubic on to the cubic again so that A', B', C' are the points obtained, then A'B'C' is also apolar to the same member S of the syzygetic system.

Let the line AB meet the cubic again in X, let O be the tangential point of T, and let XO meet the cubic again in X'. We wish to prove that $A'B'$ passes through X'. Two cubics through the *nine* points X, A, B, O, T, T, X', A', B' are the cubic S and the three-line cubic ATA', BTB', XOX'. Hence the three-line cubic XAB, OTT, $A'B'$ which passes through *eight* of these points must pass through the ninth. Thus $A'B'$ must pass through X'. Consider any other two lines XC_1C_2 and $X'C_1'C_2'$ related in the same way. Plainly a one-one correspondence exists between XC_1C_2 and $X'C_1'C_2'$; their point of intersection therefore lies on a fixed conic σ through X and X'. Now, the bipolar lines of A, B and A', B' with respect to any member of the syzygetic system of cubics are known to pass through X and X' respectively, and they are plainly corresponding members of a one-one correspondence. Hence their point of intersection must generate a conic σ' passing through X and X'. We shall now shew that the conics σ and σ' are identical. Let α_1, α_2, α_3 be the three points of contact of the tangents drawn from the tangential point of A. Then, if α_1', α_2', α_3' be the projections of α_1, α_2, α_3 through T, it is known that α_1', α_2', α_3' are the points of contact of the tangents drawn from the tangential point of A'. Furthermore, if the pencil $X[\alpha_1, \alpha_2, \alpha_3]$ meet the curve again in the points β_1, β_2, β_3, it is known that β_1, β_2, β_3 are the points of contact of the tangents drawn from the tangential point of B. Finally, if the pencil $X'[\alpha_1', \alpha_2', \alpha_3']$ meet the curve in β_1', β_2', β_3', these latter points are the points of contact of the tangents drawn from the tangential point of B', and are also the projections of β_1, β_2, β_3 through T. Hence the points of intersection of the pencil $X[\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3]$ with the corresponding members of the pencil $X'[\alpha_1\beta_1', \alpha_2'\beta_2', \alpha_3'\beta_3']$ lie on the conic σ. But, from another point of view, the polar conics of the point A with respect to the members of the syzygetic system of cubics consist of the conics passing through the points of contact of the four tangents drawn from A to the original cubic, and the three two-line conics of the system have their vertices at α_1, α_2, α_3, and are the polar conics of A with respect to those *three* members of the syzygetic system which have the given cubic as their Hessian. Consider now the first degenerate polar conic of A which has its vertex at α_1. The polar line of B with respect to this two-line conic, is evidently $X\alpha_1$. Hence the bipolar lines of A, B and of A', B' with respect to the three members of the syzygetic system which

have the given cubic as Hessian are $X\alpha_1\beta_1$, $X\alpha_2\beta_2$, $X\alpha_3\beta_3$ and $X'\alpha_1'\beta_1'$, $X'\alpha_2'\beta_2'$, $X'\alpha_3'\beta_3'$ respectively. The points of intersection of corresponding members of these two pencils lie therefore on the conic σ'. Hence the conics σ and σ' must intersect in *three* points in addition to X, X', and must therefore be identical. We have therefore established the following result:—

If A, B be two given points on a cubic curve and if A', B' be their projections through any point T on to the cubic again, then the two points C_1, C_2 which with A, B each form triangles apolar to a given member of the syzygetic pencil project through T on to the cubic again into two other points C_1', C_2' each of which along with A', B' form triangles apolar to the same syzygetic member.

We have thus demonstrated the first general theorem of the paper, and we proceed to deal with the second.

If A, B, C be a triangle inscribed in a cubic curve, which has one of its points of inflexion I situated at infinity and the line Ox as the "harmonic line" relative to I, and if S be the member of the syzygetic system of cubics to which A, B, C is apolar, then the three lines B_0C, C_0A, A_0B meet the original cubic in three collinear points F, G, H, and the line FGH intersects the tangent at I to S in a point on the "harmonic line" of I, it being given that IA, IB, IC meet the given cubic again in A_0, B_0, C_0.

As before, consider the points A, B as given, and let XC_1C_2 be a member of the pencil of lines through the point X in which the line AB meets the curve again. Let IC_1 and IC_2 meet the curve again in $C_{1,0}$ and $C_{2,0}$ respectively. Consider, first, the three collinear points defined by the intersections of B_0C_1, $C_{1,0}A$, A_0B with the curve, and secondly, by the intersections of B_0C_2, $C_{2,0}A$, A_0B with the curve. It is easily seen that the lines B_0C_1 and $C_{2,0}A$ intersect each other in a point lying on the curve, and similarly B_0C_2 meets $C_{1,0}A$ in a point on the curve. Hence the two lines thus defined are the same, and we may denote them by FGH where H is a fixed point, namely the intersection of A_0B with the cubic. Thus, plainly a one-one correspondence exists between the lines FGH and XC_1C_2, from which it is at once deduced that a one-one correspondence exists between FGH and the tangent at I to that particular member of the syzygetic pencil, with respect to which XC_1C_2 is the bipolar line of A, B. Let the above tangent at I and FGH meet the "harmonic line" of I in W and R respectively. Then a one-one correspondence exists between W and R. Let now the given cubic meet the harmonic

line of I in the points E_1, E_2, E_3. Then the tangents at I to the three cubics having the given cubic as Hessian are known to be IE_1, IE_2, IE_3. Also the bipolar lines of A, B with respect to these three members of the syzygetic pencil are as before $X\alpha_1\beta_1$, $X\alpha_2\beta_2$, $X\alpha_3\beta_3$. To obtain the line FGH corresponding to $X\alpha_1\beta_1$, we consider the intersections of the lines $B_0\alpha_1$, $\alpha_{1,0}A$, A_0B with the curve, as above described. Since A and α_1 have the same tangential point, it is evident that the line $\alpha_{1,0}A$ passes through E_1, and so for E_2 and E_3. Hence the lines FGH corresponding to IE_1, IE_2, IE_3 are respectively HE_1, HE_2, HE_3. We thus see that the one-one correspondence between W and R has *three* coincident points, proving that W and R are always one and the same point. We therefore obtain that the line FGH, corresponding to XC_1C_2, cuts the "harmonic line" of I at the point where it is met by the tangent at I to that particular member of the syzygetic pencil with respect to which XC_1C_2 is the bipolar line of A, B. Thus the second general theorem is established.

THE SINGULARITIES OF THE ALGEBRAIC TROCHOIDS

By D. M. Y. SOMMERVILLE.

[Received and Read February 14th, 1918.—Revised version Received January 11th, 1919.]

1. The degree and class of the general algebraic epi- or hypo-trochoid were given by T. Cotterill.* S. Roberts† showed that the curves possess multiple points at the circular points, but his statements regarding the composition of these singularities are not true generally, and he gives no investigation of the double points, double tangents, and inflexions. I have determined the number of the fundamental singularities for the various types of curves and have analysed the compound singularities at infinity. A brief account of the methods employed and of the results obtained may be of interest.

2. The curves are defined as follows. Two points P and Q describe circles of radii P and Q with constant angular velocities p and q respectively in *opposite* senses. Their relative path is an epi- or hypotrochoid according as p/q is positive or negative. The well-known double modes of generation by rolling one circle on another are obtained at once by considering the path of P relative to Q, and that of Q relative to P. If a and b are the radii of the fixed and rolling circles, and c the distance of the tracing point from the centre of the rolling circle, the two alternatives are

$$\left.\begin{aligned} a &= Q(1-q/p), & a' &= P(1-p/q), \\ b &= Q\,q/p, & b' &= P\,p/q, \\ c &= P, & c' &= Q. \end{aligned}\right\} \qquad (1)$$

The condition for an epi- or hypocycloid is $pP = \pm qQ$.

* *Educational Times*, November 1865.

† "Note on the Plückerian Characteristics of Epi- and Hypo-Trochoids and Allied Curves," *Proc. London Math. Soc.*, Ser. 1, Vol. 4 (1873), pp. 353–356.

p and q are taken to be integers with no common factor, and p is positive and numerically greater than q. The curve is an epi- or hypotrochoid according as q is positive or negative. P and Q are always positive.

3. The equations of the curve (path of P relative to Q) are given in parametric form

$$\left.\begin{aligned} x &= P\cos p\phi + Q\cos q\phi, \\ y &= P\sin p\phi + Q\sin q\phi, \end{aligned}\right\} \tag{2}$$

the axis of x, or initial line, being the line in which OQP are initially collinear, and $q\phi$ being the angle XOQ.

The corresponding line equations are, for the epi- and hypotrochoid,

$$\left.\begin{aligned} l &= pP\cos p\phi + qQ\cos q\phi, \\ m &= pP\sin p\phi + qQ\sin q\phi, \\ n &= -(pP^2+qQ^2) - (p+q)\,PQ\cos(p-q)\phi; \end{aligned}\right\} \tag{3a}$$

for the epicycloid,

$$\left.\begin{aligned} l &= \cos\tfrac{1}{2}(p+q)\,\phi, \\ m &= \sin\tfrac{1}{2}(p+q)\phi, \\ n &= -(P+Q)\cos\tfrac{1}{2}(p-q)\phi; \end{aligned}\right\} \tag{3b}$$

and for the hypocycloid,

$$\left.\begin{aligned} l &= \sin\tfrac{1}{2}(p+q)\phi, \\ m &= -\cos\tfrac{1}{2}(p+q)\,\phi, \\ n &= (P-Q)\sin\tfrac{1}{2}(p-q)\,\phi. \end{aligned}\right\} \tag{3c}$$

4. Algebraic freedom equations are obtained by taking as triangle of reference the triangle OIJ, I and J being the circular points, so that $X \equiv x+iy$, $Y \equiv x-iy$, $Z \equiv 1$. Putting $e^{i\phi} = u$, the equations are then*

$$\left.\begin{aligned} X &= Pu^{2p} + Qu^{p+q}, \\ Y &= P \quad\; + Qu^{p-q}, \\ Z &= u^p. \end{aligned}\right\} \tag{4}$$

* These equations have been used with advantage by several writers, *e.g.* F. Morley, "On the Epicycloid," *Amer. Jour. of Math.*, Vol. 13 (1891), pp. 179–184.

The corresponding line equations, in which

$$L \equiv \tfrac{1}{2}(l-im), \quad M \equiv \tfrac{1}{2}(l+im), \quad N \equiv n,$$

are

$$\left.\begin{aligned} L &= pPu^{-p}+qQu^{-q}, \\ M &= pPu^{p}\ +qQu^{q}, \\ N &= -(p+q)PQ(u^{p-q}+u^{-p+q})-2(pP^2+qQ^2), \end{aligned}\right\} \tag{5}$$

which have to be adjusted to the proper form, free from negative powers, according as q is positive or negative.

For an epicycloid $pP = qQ$ and these equations reduce to

$$L = 1, \quad M = u^{p+q}, \quad N = -(P+Q)(u^p+u^q); \tag{5a}$$

for a hypocycloid $pP = -qQ$, and the equations reduce to

$$L = -u^{-q}, \quad M = u^p, \quad N = (P-Q)(u^{p-q}-1). \tag{5b}$$

5. From these equations the degree and class are immediately obtained. Further, since the curves are unicursal, the fundamental singularities can be obtained from Plücker's equations, which give

$$\delta = \tfrac{1}{2}(n-1)(n-6)+m, \qquad \tau = \tfrac{1}{2}(m-1)(m-6)+n,$$

$$\kappa = 2(n-1)-m, \qquad \iota = 2(m-1)-n.$$

Hence we have the results

	Epitrochoid.	Hypotrochoid.	Epicycloid.	Hypocycloid.
n	$2p$	$2p$	$2p$	$2p$
m	$2p$	$2(p-q)$	$p+q$	$p-q$
δ	$(p-1)(2p-3)$	$(p-1)(2p-3)-2q$	$(p-1)(2p-3)-(p-q)$	$(p-1)(2p-3)-(p+q)$
κ	$2(p-1)$	$2(p+q-1)$	$2(p-1)+(p-q)$	$2(p-1)+(p+q)$
τ	$(p-1)(2p-3)$	$(p-q-1)\{2(p-q)-3\}+2q$	$\tfrac{1}{2}(p+q-1)(p+q-2)-2(q-1)$	$\tfrac{1}{2}(p-q-1)(p-q-2)+2(q+1)$
ι	$2(p-1)$	$2(p-2q-1)$	$2(q-1)$	$-2(q+1)$

6. If the equations (4) and (5) are made homogeneous by introducing another parameter w, or putting u/w instead of u, it is obvious that the

only points at infinity are the circular points $I \equiv (0, 1, 0)$ and $J \equiv (1, 0, 0)$ corresponding to the parameters $u = 0$ and $w = 0$.

For the epitrochoid and epicycloid these are multiple points of order p, the tangents at each, $X = 0$ and $Y = 0$ respectively, being multiple of order q, and corresponding to the parameters $u = 0$ and $w = 0$.

For the hypotrochoid and hypocycloid the circular points are multiple only of order $p+q$; the tangent at each, $Z = 0$, is multiple of order $-2q$, and corresponds to the parameters $u = 0$ and $w = 0$. If $p+q = 1$, the circular points are no longer multiple, but the line at infinity is still a multiple tangent touching at I and J.

An ordinary multiple point of order p is composed of $\frac{1}{2}p\,(p-1)$ double points or cusps. Roberts assumed that this is so in the present case, but it appears that this is only true when $q = \pm 1$, *i.e.* for curves of one revolution.

7. The composition of the singularities at infinity will be determined when we know the number of the finite singularities.

The method which is employed to determine these is as follows. Consider a curve which is represented by the freedom equations

$$x = f(u), \quad y = g(u), \quad z = h(u).$$

A double point has two parametric values. If these are u and v they are determined by the equations

$$\frac{f(u)}{f(v)} = \frac{g(u)}{g(v)} = \frac{h(u)}{h(v)}.$$

If these are made homogeneous by the introduction of w, they may be represented by two curves in the homogeneous coordinates u, v, w, with triangle of reference, say ABC. As the equations are satisfied identically by $u = v$, in general each curve contains the straight line $u = v$, which must be removed. If there are still intersections on this line these will correspond to double points for which the two parameters become equal, *i.e.* cusps, or, perhaps, higher singularities. Care must be taken to determine whether the curves touch on the line $u = v$, and if so to what degree of contact. In the present case we require to find the finite double points, and therefore we reject any intersections at A, B, or C.

An exactly similar method can be applied to find the double tangents, using the freedom equations in line coordinates. In this case intersections

on the line $u = v$ may correspond to inflexions or points of undulation, &c.

To find the cusps and inflexions we may proceed thus. If u is the parameter of a stationary point or tangent, the line $lx+my+nz = 0$ will cut the curve in three coincident points. Hence writing

$$F(u) \equiv lf(u)+mg(u)+nh(u),$$

we must have the three equations

$$F(u) = 0, \quad F'(u) = 0, \quad F''(u) = 0.$$

Eliminating l, m, n between these three equations we obtain an equation in u which gives the parameters of the cusps and points of inflexion.

8. The following are the results.

Finite Singularities.

	Epitrochoid.	Hypotrochoid.	Epicycloid.	Hypocycloid.
δ	$(p-1)(p-q)$	$(p-1)(p-q)$	$(p-2)(p-q)$	$(p-2)(p-q)$
κ	0	0	$p-q$	$p-q$
τ	$(p-q)(2p+q-4)$	$(p-q)(2p-q-4)$	$\frac{1}{2}(p-q)(p+q-1)$	$\frac{1}{2}(p-q)(p+q-1)$
ι	$2(p-q)$	$2(p-q)$	0	0

9. *Composition of the Singularities at I and J.*

Epitrochoid and Epicycloid.

Each of the circular points is a multiple point of order p, with a multiple tangent, passing through O, of order q, composed of $\frac{1}{2}(p-1)(p+q-3)$ double points, $p-1$ cusps, $\frac{1}{2}(q-1)(p+q-3)$ double tangents, and $q-1$ inflexions.

Hypotrochoid and Hypocycloid.

Each of the circular points is a multiple point of order $p+q$, with the line at infinity as a multiple tangent of order $-2q$. Together they contain $(p-3)(p+q-1)$ double points, $2(p+q-1)$ cusps, $-2(q+1)$ inflexions, and $q^2-(q+1)(p-3)$ double tangents.

Curves of one Revolution.

If $q = 1$, we have an epitrochoid of one revolution; there are no double tangents or inflexions at infinity, and the circular points consist each of $\frac{1}{2}(p-1)(p-2)$ double points and $p-1$ cusps.

If $q = -1$, we have a hypotrochoid of one revolution; there are no inflexions at infinity; the circular points consist each of $\frac{1}{2}(p-2)(p-3)$ double points and $p-2$ cusps, and the line at infinity is a double tangent.

10. *Double points.*

The parameters ϕ, ϕ' of a double point are determined by the equations

$$\left.\begin{aligned}\phi &= \frac{k\pi}{p-q} + \psi,\\ \phi' &= \frac{k\pi}{p-q} - \psi,\end{aligned}\right\} \qquad P \sin p\psi = (-)^{k+1} Q \sin q\psi,$$

k being any integer, and its polar coordinates are

$$\theta = \frac{kp\pi}{p-q}, \qquad r = P \cos p\psi + (-)^k Q \cos q\psi.$$

There are $p-q$ "primary nodes", those which become cusps in the epi- and hypocycloid. All the nodes lie on $p-q$ "nodal lines" through O. If $p-q$ is odd there is one primary node on each of these, and if $p-q$ is even the primary nodes lie in pairs on half the nodal lines.

Excluding acnodes there are at least $(|q|-1)(p-q)$ real nodes.

If $P = Q$, the origin is a multiple point of order $p-q$ consisting of $\frac{1}{2}(p-q)(p-q-1)$ double points.

11. *Cusps.*

For the epi- and hypocycloid the parameters of the cusps are

$$\phi = (2k+1)\pi/(p-q).$$

12. *Double tangents.*

Epi- and Hypocycloid.

The parameters ϕ, ϕ' of a double tangent are

$$\phi = \frac{k\pi}{p+q} + \frac{k'\pi}{p-q},$$

$$\phi' = -\frac{k\pi}{p+q} + \frac{k'\pi}{p-q},$$

where $k = 1, 2, \ldots, (p+q-1)$, and $k' = 1, 2, \ldots, 2(p-q)$; for the epicycloid k and k' are either both odd or both even, for the hypocycloid one is odd and the other even.

Epi- and Hypotrochoid.

Of the $(p-q)(2p+|q|-4)$ double tangents $(p-1)(p-q)$ are "direct," each being perpendicular to a nodal line, and the number of "transverse" double tangents is $(p-q)(p+|q|-3)$.

The parameters of the direct double tangents are determined by the equations

$$\left.\begin{aligned} \phi &= \frac{k\pi}{p-q} + \psi, \\ \phi' &= \frac{k\pi}{p-q} - \psi, \end{aligned}\right\} \qquad pP \sin p\psi = (-)^{k+1} qQ \sin q\psi.$$

The polar coordinates of the foot of the perpendicular from the origin are

$$\theta = \frac{kp\pi}{p-q}, \qquad r = \frac{pP^2 + qQ^2 + (-)^k (p+q)\, PQ \cos(p-q)\,\psi}{pP \cos p\psi + (-)^k\, qQ \cos q\psi}.$$

The parameters u, u' of the transverse double tangents are determined by the equations

$$(p-q)(K^2-1)\, u^p = (p+q)(K + u^{p-q})(Ku'^{p-q} + 1)\, u'^q,$$

$$(p-q)(K^2-1)\, u'^p = (p+q)(K + u'^{p-q})(Ku^{p-q} + 1)\, u^q,$$

where $$K = pP/qQ.$$

13. *Inflexions.*

The parameters of the inflexions are determined by the equation

$$\cos(p-q)\phi = -\frac{p^3P^2 + q^3Q^2}{pq(p+q)PQ} = -1 - \frac{(pP-qQ)(p^2P-q^2Q)}{pq\,(p+q)\,PQ}.$$

The conditions that the inflexions should be real are $pP < |qQ|$ and $p^2P > q^2Q$; the former condition requires that the primary nodes should be acnodes. If $p^2P = q^2Q$, the inflexions coalesce in pairs, absorbing also a double tangent, and we get points of undulation.

The points of inflexion all lie on the circle

$$r^2 = (p-q)\{-P^2p(2p+q)+Q^2q(p+2q)\}/pq(p+q),$$

and the inflexional tangents all touch the circle

$$r^2 = (p^2-q^2)(-P^2p^2+Q^2q^2)/p^2q^2.$$

THE VIBRATIONS AND STABILITY OF A ROTATING CYLINDER

By F. B. PIDDUCK.

[Read June 12th, 1919.]

1. In a paper on the "Stability of Rotating Shafts"* I attempted to discuss the vibrations of a circular cylinder, rotating about its axis of figure, from the general equations of elasticity, following Pochhammer's method† of simple exact solutions. Some time ago, Prof. W. McF. Orr, objecting to the conclusion that the torsional vibrations might become exponentially unstable, pointed out that the fundamental equations could not possibly be correct, as they were not satisfied by a rigid body displacement consisting of a small rotation round the axis. A note of acknowledgment was recently inserted in the *Proceedings*.‡ In subsequent correspondence Prof. Orr, modifying to some extent his opinions as first expressed, considered that to be quite accurate it would be necessary to allow for finite stresses in the steady motion; without however undertaking any formal investigation. Subsequent study has convinced me of the necessity of proceeding on the lines initiated by Southwell,§ who was the first to give a satisfactory treatment of elastic stability on the basis of the general equations. In addition to the neglect of these principles, a serious error was committed in my paper, namely that the acceleration was calculated for a displaced point and the stress-resultant for a point momentarily occupying the undisplaced position (r, θ, z), thus producing the discrepancy pointed out by Prof. Orr. The application of Southwell's principles to the present problem appears more worth the trouble in that the stability of the steady rotary motion above the critical speed, on any rigorous theory, has not been decided in the meantime. Previous treatments have, in effect, considered the rod as constrained to move in a plane

* *Proceedings*, Ser. 2, Vol. 9 (1910), p. 352.
† L. Pochhammer, *Crelle's Journal*, Vol. 81 (1876), p. 324.
‡ *Proceedings*, Ser. 2, Vol. 16 (1918), p. lviii.
§ R. V. Southwell, *Phil. Trans.*, (A), Vol. 213 (1913), p. 187.

turning with constant angular velocity.* Such considerations (of statical centrifugal force) are, of course, only applicable to the cases of whirling or steady rotation; but Chree† has once applied them to the case of an angular velocity variable with the time.

2. To fix ideas, the steady motion will be taken to be one of plane strain without longitudinal extension. As usual in problems of this type, no attempt will be made to satisfy exact boundary conditions of any of the planes parallel to $z = 0$. Adopting Southwell's definitions of stress and his stress-strain relations, there is an exact solution of the problem of steady motion even when the strain is finite; but to the order of approximation used it is sufficient to take the principal extensions e_1, e_2, e_3 to be given by the ordinary formulæ

$$e_1 = \frac{\rho\omega^2\{(2\lambda+3\mu)a^2-3(\lambda+\mu)r^2\}}{8(\lambda+\mu)(\lambda+2\mu)}, \qquad e_2 = \frac{\rho\omega^2\{(2\lambda+3\mu)a^2-(\lambda+\mu)r^2\}}{8(\lambda+\mu)(\lambda+2\mu)},$$

$$e_3 = 0.$$

Now consider a small motion superposed on the state of steady rotation, in which (u_r, u_θ, u_z) is the additional displacement. The final point corresponding to (r, θ, z) being $(\bar{r}, \bar{\theta}, \bar{z})$, we have

$$\bar{r} = r(1+e_2)+u_r, \qquad \bar{\theta} = \theta+u_\theta/r(1+e_2), \qquad \bar{z} = z+u_z.$$

The components of acceleration, parallel to radial, tangential and axial lines in the final position, are

$$\bar{f}_r = \frac{\partial^2 u_r}{\partial t^2} - 2\omega\frac{\partial u_\theta}{\partial t} - \omega^2\{r(1+e_2)+u_r\}, \qquad \bar{f}_\theta = \frac{\partial^2 u_\theta}{\partial t^2} + 2\omega\frac{\partial u_r}{\partial t}, \qquad \bar{f}_z = \frac{\partial^2 u_z}{\partial t^2},$$

and the density at the displaced point is given by $\bar{\rho}\,\bar{r}\,d\bar{r}\,d\bar{\theta}\,d\bar{z} = \rho r\,dr\,d\theta\,dz$. Since

$$\frac{\partial(\bar{r}\bar{\theta}\bar{z})}{\partial(r\theta z)} = 1+e_1+\frac{\partial u_r}{\partial r}+\frac{1+e_1}{1+e_2}\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+(1+e_1)\frac{\partial u_z}{\partial z},$$

to the first order in (u_r, u_θ, u_z), we find

$$\bar{\rho} = \frac{\rho}{(1+e_1)(1+e_2)}\left[1-(1-e_1)\frac{\partial u_r}{\partial r}-(1-e_2)\frac{u_r}{r}-(1-e_2)\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{\partial u_z}{\partial z}\right],$$

* Cf. for example, C. Chree, *Phil. Mag.*, [6], Vol. 7 (1904), p. 504; A. Fage, *Engineering*, Vol. 104 (1917), p. 53; Sir A. G. Greenhill, *Engineering*, Vol. 105 (1918), p. 273.

† C. Chree, *Trans. Camb. Phil. Soc.*, Vol. 14 (1887), p. 357.

where we now begin to neglect terms in which u_r, u_θ, u_z are multiplied by non-linear terms in e_1, e_2. In forming the product $\bar{\rho}_r \bar{f}_r$ we also neglect terms in which u_r, u_θ, u_z or their derivates are multiplied by products of $\rho\omega^2$ and e_1 or e_2; that is, terms of order $\rho\omega^2 eu$. Thus

$$\bar{\rho}\bar{f}_r = -\frac{\rho r\omega^2}{1+e_1} + \rho(1-e_1-e_2)\left[\frac{\partial^2 u_r}{\partial t^2} - 2\omega\frac{\partial u_\theta}{\partial t}\right] + \rho r\omega^2\left[\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right],$$

$$\bar{\rho}\bar{f}_\theta = \rho(1-e_1-e_2)\left[\frac{\partial^2 u_\theta}{\partial t^2} + 2\omega\frac{\partial u_r}{\partial t}\right], \qquad \bar{\rho}\bar{f}_z = \rho(1-e_1-e_2)\frac{\partial^2 u_z}{\partial t^2}.$$

The stress-resultants are calculated by a simple modification of the method of Southwell.* Since

$$\widehat{rr} = (\lambda+2\mu)e_1+\lambda e_2, \qquad \widehat{\theta\theta} = \lambda e_1+(\lambda+2\mu)e_2, \qquad \widehat{zz} = \lambda e_1+\lambda e_2,$$

we have, to the first order in e_1 and e_2,

$$\frac{(1+e_2)\widehat{\theta\theta}-(1+e_3)\widehat{zz}}{(1+e_2)^2-(1+e_3)^2} = \mu+\frac{\lambda e_1+(\lambda+\mu)e_2}{2},$$

$$\frac{(1+e_3)\widehat{zz}-(1+e_1)\widehat{rr}}{(1+e_3)^2-(1+e_1)^2} = \mu+\frac{(\lambda+\mu)e_1+\lambda e_2}{2},$$

$$\frac{(1+e_1)\widehat{rr}-(1+e_2)\widehat{\theta\theta}}{(1+e_1)^2-(1+e_2)^2} = \mu+\frac{(\lambda+\mu)e_1+(\lambda+\mu)e_2}{2}.$$

Hence, to the required order of approximation,

$$\widehat{rr} = \frac{(\lambda+2\mu)e_1+\lambda e_2}{1+e_2} - [(\lambda+2\mu)e_1+\lambda e_2]\left[\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right]$$
$$+(1-e_2)\left[(\lambda+2\mu)\frac{\partial u_r}{\partial r}+\lambda\frac{u_r}{r}+\frac{\lambda}{r}\frac{\partial u_\theta}{\partial\theta}+\lambda\frac{\partial u_z}{\partial z}\right],$$

$$\widehat{\theta\theta} = \frac{\lambda e_1+(\lambda+2\mu)e_2}{1+e_1} - [\lambda e_1+(\lambda+2\mu)e_2]\left[\frac{\partial u_z}{\partial z}+\frac{\partial u_r}{\partial r}\right]$$
$$+(1-e_1)\left[(\lambda+2\mu)\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\lambda\frac{\partial u_z}{\partial z}+\lambda\frac{\partial u_r}{\partial r}\right],$$

$$\widehat{zz} = \frac{\lambda e_1+\lambda e_2}{(1+e_1)(1+e_2)} - [\lambda e_1+\lambda e_2]\left[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right]$$
$$+(1-e_1-e_2)\left[(\lambda+2\mu)\frac{\partial u_z}{\partial z}+\lambda\frac{\partial u_r}{\partial r}+\lambda\frac{u_r}{r}+\frac{\lambda}{r}\frac{\partial u_\theta}{\partial\theta}\right],$$

---

* *Loc. cit.*, pp. 203–7.

$$\widehat{\theta z} = \left[\mu + \frac{\lambda-2\mu}{2} e_1 + \frac{\lambda-\mu}{2} e_2\right] \frac{\partial u_\theta}{\partial z} + \left[\mu + \frac{\lambda-2\mu}{2} e_1 + \frac{\lambda+\mu}{2} e_2\right] \frac{1}{r} \frac{\partial u_z}{\partial \theta},$$

$$\widehat{zr} = \left[\mu + \frac{\lambda-\mu}{2} e_1 + \frac{\lambda-2\mu}{2} e_2\right] \frac{\partial u_r}{\partial z} + \left[\mu + \frac{\lambda+\mu}{2} e_1 + \frac{\lambda-2\mu}{2} e_2\right] \frac{\partial u_z}{\partial r},$$

$$\widehat{r\theta} = \left[\mu + \frac{\lambda-\mu}{2} e_1 + \frac{\lambda+\mu}{2} e_2\right] \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \left[\mu + \frac{\lambda+3\mu}{2} e_1 + \frac{\lambda-3\mu}{2} e_2\right] \frac{u_\theta}{r}$$
$$+ \left[\mu + \frac{\lambda+\mu}{2} e_1 + \frac{\lambda-\mu}{2} e_2\right] \frac{\partial u_\theta}{\partial r}.$$

Also

$$\frac{\partial}{\partial \bar{r}} = \frac{1}{1+e_1} \frac{\partial}{\partial r} - (1-2e_1) \frac{\partial u_r}{\partial r} \frac{\partial}{\partial r},$$

$$\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{\theta}} = \frac{1}{1+e_2} \frac{1}{r} \frac{\partial}{\partial \theta} - (1-e_1-e_2) \frac{1}{r} \frac{\partial u_r}{\partial \theta} \frac{\partial}{\partial r},$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} - (1-e_1) \frac{\partial u_r}{\partial z} \frac{\partial}{\partial r},$$

$$\frac{1}{\bar{r}} = \frac{1}{1+e_2} \frac{1}{r} - (1-2e_2) \frac{u_r}{r^2},$$

so that we are in a position to write down, in explicit form, the equations of motion

$$\frac{\partial \widehat{\bar{r}\bar{r}}}{\partial \bar{r}} + \frac{1}{\bar{r}} \frac{\partial \overline{r\theta}}{\partial \bar{\theta}} + \frac{\partial \widehat{\bar{z}\bar{r}}}{\partial \bar{z}} + \frac{\widehat{\bar{r}\bar{r}} - \widehat{\bar{\theta}\bar{\theta}}}{\bar{r}} = \bar{\rho}\bar{f}_r,$$

$$\frac{\partial \widehat{\bar{r}\bar{\theta}}}{\partial \bar{r}} + \frac{1}{\bar{r}} \frac{\partial \widehat{\bar{\theta}\bar{\theta}}}{\partial \bar{\theta}} + \frac{\partial \widehat{\bar{\theta}\bar{z}}}{\partial \bar{z}} + 2\frac{\widehat{\bar{r}\bar{\theta}}}{\bar{r}} = \bar{\rho}\bar{f}_\theta,$$

$$\frac{\partial \widehat{\bar{z}\bar{r}}}{\partial \bar{r}} + \frac{1}{\bar{r}} \frac{\partial \widehat{\bar{\theta}\bar{z}}}{\partial \bar{\theta}} + \frac{\partial \widehat{\bar{z}\bar{z}}}{\partial \bar{z}} + \frac{\widehat{\bar{z}\bar{r}}}{\bar{r}} = \bar{\rho}\bar{f}_z.$$

In forming the final expressions we neglect, as before, terms in u_r, u_θ, u_z which involve e_1^2, $e_1 e_2$, etc. The first terms neglected on the left are of the order of $\mu e^2 u/r^2$, or similar terms such as $(\lambda+2\mu)e^2 \partial^2 u_r/\partial r^2$, etc. These are however comparable with $\rho\omega^2 e u$, the first terms neglected in forming the product $\bar{\rho}\bar{f}_r$, so that terms of the same order are being retained on both sides.

The finite term in the equations of motion gives, of course, the exact

equation for the steady displacement, namely

$$\frac{du^2}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}+\frac{\rho\omega^2}{\lambda+2\mu}(u+r)=0,$$

remembering that $e_1=du/dr$ and $e_2=u/r$.

The differential equations of the relative motion are simplified by multiplying by $1+e_1+e_2$, when they become

$$\begin{aligned}(\lambda+2\mu)\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]&+\left[\mu+\frac{\lambda+\mu}{2}e_1+\frac{\lambda+\mu}{2}e_2\right]\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}\\&+\left[\mu+\frac{\lambda+\mu}{2}e_1+\frac{\lambda}{2}e_2\right]\frac{\partial^2 u_r}{\partial z^2}-\rho\frac{\partial^2 u_r}{\partial t^2}+\rho\omega^2 u_r\\&+(\lambda+\mu)(1-\tfrac{1}{2}e_1-\tfrac{1}{2}e_2)\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\,\partial\theta}\\&-\left[\lambda+3\mu+\frac{\lambda+\mu}{2}e_1+\frac{\lambda+\mu}{2}e_2\right]\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+2\rho\omega\frac{\partial u_\theta}{\partial t}\\&+\left[\lambda+\mu-\frac{\lambda+\mu}{2}e_1-\frac{\lambda}{2}e_2\right]\frac{\partial^2 u_z}{\partial r\,\partial z}=0,\end{aligned}$$

$$\begin{aligned}(\lambda+\mu)(1-\tfrac{1}{2}e_1-\tfrac{1}{2}e_2)\frac{1}{r}\frac{\partial^2 u_r}{\partial r\,\partial\theta}&\\+\left[\lambda+3\mu+\frac{\lambda+3\mu}{2}e_1+\frac{\lambda-\mu}{2}e_2-\frac{\lambda+\mu}{2}r\frac{de_1}{dr}-\frac{\lambda+3\mu}{2}r\frac{de_2}{dr}\right]\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}&\\-2\rho\omega\frac{\partial u_r}{\partial t}+\left[\mu+\frac{\lambda+\mu}{2}e_1+\frac{\lambda+\mu}{2}e_2\right]\frac{\partial^2 u_\theta}{\partial r^2}&\\+\left[\mu+\frac{\lambda+3\mu}{2}e_1+\frac{\lambda-\mu}{2}e_2+\frac{\lambda+\mu}{2}r\frac{de_1}{dr}+\frac{\lambda-\mu}{2}r\frac{de_2}{dr}\right]\frac{1}{r}\frac{\partial u_\theta}{\partial r}&\\+(\lambda+2\mu)\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\left[\mu+\frac{\lambda}{2}e_1+\frac{\lambda+\mu}{2}e_2\right]\frac{\partial^2 u_\theta}{\partial z^2}&\\-\left[\mu+\frac{\lambda+7\mu}{2}e_1+\frac{\lambda-5\mu}{2}e_2+\frac{\lambda+3\mu}{2}r\frac{de_1}{dr}+\frac{\lambda-3\mu}{2}r\frac{de_2}{dr}\right]\frac{u_\theta}{r^2}&\\-\rho\frac{\partial^2 u_\theta}{\partial t^2}+\left[\lambda+\mu-\frac{\lambda}{2}e_1-\frac{\lambda+\mu}{2}e_2\right]\frac{1}{r}\frac{\partial^2 u_z}{\partial\theta\,\partial z}=0,&\end{aligned}$$

$$\left[\lambda+\mu-\frac{\lambda+\mu}{2}e_1-\frac{\lambda}{2}e_2\right]\frac{\partial^2 u_r}{\partial r\partial z}$$

$$+\left[\lambda+\mu-\frac{\lambda-\mu}{2}e_1-\frac{\lambda+2\mu}{2}e_2-\frac{\lambda+\mu}{2}r\frac{de_1}{dr}-\frac{\lambda+2\mu}{2}r\frac{de_2}{dr}\right]\frac{1}{r}\frac{\partial u_r}{\partial z}$$

$$+\left[\lambda+\mu-\frac{\lambda}{2}e_1-\frac{\lambda+\mu}{2}e_2\right]\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial z}+\left[\mu+\frac{\lambda+\mu}{2}e_1+\frac{\lambda}{2}e_2\right]\frac{\partial^2 u_z}{\partial r^2}$$

$$+\left[\mu+\frac{\lambda+3\mu}{2}e_1+\frac{\lambda-2\mu}{2}e_2+\frac{\lambda+\mu}{2}r\frac{de_1}{dr}+\frac{\lambda-2\mu}{2}r\frac{de_2}{dr}\right]\frac{1}{r}\frac{\partial u_z}{\partial r}$$

$$+\left[\mu+\frac{\lambda}{2}e_1+\frac{\lambda+\mu}{2}e_2\right]\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+(\lambda+2\mu)\frac{\partial^2 u_z}{\partial z^2}-\rho\frac{\partial^2 u_z}{\partial t^2}=0.$$

In these equations (r, θ, z) is the position of the point in the *unstrained* state, *i.e.* before rotation. The boundary conditions at the cylindrical surface $r = a$ are*

$$(\lambda+2u)\frac{\partial u_r}{\partial r}+\lambda\frac{u_r}{r}+\frac{\lambda}{r}\frac{\partial u_\theta}{\partial\theta}+\lambda\frac{\partial u_z}{\partial z}=0,$$

$$(1+e_1)\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+(1+e_2)\frac{\partial u_\theta}{\partial r}=0,$$

$$(1+e_1)\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}=0.$$

3. Putting $(u_r, u_\theta, u_z) = (R_1, R_2, R_3)\exp i(n\theta+\gamma z+pt)$, where R_1, R_2, R_3 are functions of r, we have three ordinary differential equations for R_1, R_2, R_3, and three boundary conditions at $r = a$. There does not appear to be any general solution in Bessel's functions when the e terms are included, but following Southwell† the equation can be solved approximately for thin rods by expanding the functions in ascending powers of r. It is convenient to write

$$\tfrac{1}{2}e_1 = Pa^2-3Qr^2, \quad \tfrac{1}{2}e_2 = Pa^2-Qr^2,$$

where $P = (2\lambda+3u)\rho\omega^2/16(\lambda+\mu)(\lambda+2\mu)$, $Q = \rho\omega^2/16(\lambda+2\mu)$.

The case that particularly interests us is that of quasi-transverse vibra-

* Southwell, *loc. cit.*, p. 213.

† The method is virtually Pochhammer's.

tions (including whirling), characterised by $n = 1$. Assume

$$R_1 = a_0 + a_1 r^2 + a_2 r^4,$$

$$R_2 = ib_0 + ib_1 r^2 + ib_2 r^4,$$

$$R_3 = ic_0 r + ic_1 r^3 + ic_2 r^5,$$

and substitute in the equations of motion. The necessary and sufficient condition for the stresses to remain finite at $r = 0$ is $b_0 = a_0$, thus leaving eight ratios to be eliminated. Neglecting higher powers than r^3 in the equations of motion and equating coefficients we have the following six equations (two being identical):—

$$\{\rho p^2 - 2\rho\omega p + \rho\omega^2 - \mu\gamma^2 - (2\lambda+\mu) P\gamma^2 a^2\} a_0 + \{3\lambda + 5\mu - (2\lambda+2\mu) Pa^2\} a_1 + \{-\lambda + \mu + (6\lambda+6\mu) Pa^2\} b_1 + \{-\lambda\gamma - \mu\gamma + (2\lambda+\mu) P\gamma a^2\} c_0 = 0,$$

$$(4\lambda+3\mu) Q\gamma^2 a_0 + \{\rho p^2 + \rho\omega^2 - \mu\gamma^2 + (4\lambda+4\mu) Q - (2\lambda+\mu) P\gamma^2 a^2\} a_1 + \{15\lambda + 29\mu - (2\lambda+2\mu) Pa^2\} a_2 + \{-2\rho\omega p - (12\lambda+12\mu) Q\} b_1 + \{-3\lambda - \mu + (10\lambda+10\mu) Pa^2\} b_2 - (4\lambda+3\mu) Q\gamma c_0 + \{-3\lambda\gamma - 3\mu\gamma + (6\lambda+3\mu) P\gamma a^2\} c_1 = 0,$$

$$\{\rho p^2 - 2\rho\omega p - \mu\gamma^2 + (16\lambda+32\mu) Q - (2\lambda+\mu) P\gamma^2 a^2\} a_0 + \{3\lambda + 5\mu - (2\lambda+2\mu) Pa^2\} a_1 + \{-\lambda + \mu + (6\lambda+6\mu) Pa^2\} b_1 + \{-\lambda\gamma - \mu\gamma + (2\lambda+\mu) P\gamma a^2\} c_0 = 0,$$

$$(4\lambda+\mu) Q\gamma^2 a_0 + \{-2\rho\omega p + (12\lambda+12\mu) Q\} a_1 + \{5\lambda + 7\mu - (6\lambda+6\mu) Pa^2\} a_2 + \{\rho p^2 - \mu\gamma^2 - (20\lambda+4\mu) Q - (2\lambda+\mu) P\gamma^2 a^2\} b_1 + \{-\lambda + 13\mu + (30\lambda+30\mu) Pa^2\} b_2 - (4\lambda+\mu) Q\gamma c_0 + \{-\lambda\gamma - \mu\gamma + (2\lambda+\mu) P\gamma a^2\} c_1 = 0,$$

$$(8\lambda+8\mu) Q\gamma a_0 + \{3\lambda\gamma + 3\mu\gamma) - (6\lambda+3\mu) P\gamma a^2\} a_1 + \{-\lambda\gamma - \mu\gamma + (2\lambda+\mu) P\gamma a^2\} b_1 + \{\rho p^2 - (\lambda+2\mu)\gamma^2 - (8\lambda+8\mu) Q\} c + \{8\mu + (16\lambda+8\mu) Pa^2\} c_1 = 0,$$

$$(20\lambda+15\mu) Q\gamma a_1 + \{5\lambda\gamma + 5\mu\gamma - (10\lambda+5\mu) P\gamma a^2\} a_2 - (4\lambda+\mu) Q\gamma b_1 + \{-\lambda\gamma - \mu\gamma + (2\lambda+\mu) P\gamma a^2\} b_2 + \{\rho p^2 - (\lambda+2\mu)\gamma^2 - (56\lambda+44\mu) Q\} c_1 + \{24\mu + (48\lambda+24\mu) Pa^2\} c_2 = 0.$$

The boundary conditions, correct to terms in a^2, are

$$(3\lambda+4\mu)a_1+(5\lambda+8\mu)a^2a_2-\lambda b_1-\lambda a^2b_2-\lambda\gamma c_0-\lambda\gamma a^2c_1=0,$$

$$(1+2Pa^2-6Qa^2)a_1+a^2a_2+(1+2Pa^2+2Qa^2)b_1+3a^2b_2=0,$$

$$(\gamma+2P\gamma a^2-6Q\gamma a^2)\,a_0+\gamma a^2a_1+c_0+3a^2c_1=0.$$

Hence the periods are given by an eighth order determinantal equation, which reduces at once to one of the seventh order, since only one of the equations involves c_2. After some simplification by rows and columns the determinant takes the form given on the next page.

In expanding the determinant it is to be observed that the term independent of a is

$$4\times768\mu^2(\lambda+\mu)(\lambda+2\mu)(\rho p^2-2\rho\omega p+\rho\omega^2),$$

so that we may put $p=\omega$ in the terms involving a^2. Evaluating the determinant and rejecting the factor $768\mu^2$, we find the period equation in the form

$$4(\lambda+\mu)(\lambda+2\mu)(\rho p^2-2\rho\omega p+\rho\omega^2)-(3\lambda+2\mu)(\lambda+2\mu)\,\mu\gamma^4a^2$$
$$-\lambda(\lambda+2\mu)\,\rho\omega^2\gamma^2a^2=0,$$

or
$$\rho(p-\omega)^2=\tfrac{1}{4}E\gamma^4a^2+\tfrac{1}{2}\sigma\rho\omega^2\gamma^2a^2.$$

Confining our attention to modes of vibration in which γa is small compared with 1, the whirling speed Ω for wave-length $2\pi/\gamma$ is given by

$$\rho\Omega^2=\tfrac{1}{4}E\gamma^4a^2;$$

and we have simply
$$p=\omega\pm\Omega.$$

The steady rotary motion is therefore stable above the critical speed.

The character of the vibration is also peculiarly simple, being given to a first approximation by the relation $b_0=a_0$. Hence a typical small vibration is represented by

$$u_r=a_0\cos(\theta+\gamma z+pt),\qquad u_\theta=-a_0\sin(\theta+\gamma z+pt).$$

Relative to axes *fixed in space*, the geometrical centre of a section of the shaft describes a circle with angular velocity $\pm\Omega$, while the section rotates round this centre with angular velocity ω. This perhaps gives the most vivid idea of the continuity of the phenomena below and above the critical speed, and of the possibility of steady whirling motion when $\omega=\Omega$. The path of any particle in space, in this simple type of vibration, is in general an epi- or hypo-trochoid. Superposing waves travelling

0,	$-8Qa^2$,	a^2,	$1+2Pa^2+2Qa^2$,	0,	0,	0
$2\gamma+2P\gamma a^2-6Q\gamma a^2$,	γa^2,	0,	0,	0,	1,	$3a^2$
$-\lambda\gamma^2$,	$4\lambda+4\mu$,	$(3\lambda+5\mu)\,a^2$,	$-\lambda$,	$-(2\lambda+3\mu)\,a^2$,	$-\lambda\gamma$,	$-\lambda\gamma a^2$
$\rho p^2-2\rho\omega p+\rho\omega^2$ $-(\lambda+2\mu)\,\gamma^2$,	$4\lambda+4\mu-(8\lambda+8\mu)\,Pa^2$,	0,	$-\lambda+\mu+(6\lambda+6\mu)\,Pa^2$,	0,	$-\lambda\gamma-\mu\gamma+(2\lambda+\mu)P\gamma a^2$,	0
0,	$4\rho p^2+8\rho\omega p+\rho\omega^2-4\mu\gamma^2$ $-(80\lambda+32\mu)Q-(8\lambda+4\mu)P\gamma^2a^2$,	0,	$-3\rho p^2-2\rho\omega p+3\mu\gamma^2$ $+48\lambda Q+(6\lambda+3\mu)\,P\gamma^2a^2$,	-8μ $-(16\lambda+16\mu)Pa^2$,	$8\lambda Q\gamma$,	0
0,	$\rho\omega^2+(48\lambda+32\mu)\,Q$,	$12\lambda+24\mu$,	$\rho p^2-2\rho\omega p-\mu\gamma^2$ $-(32\lambda+16\mu)Q-(2\lambda+\mu)P\gamma^2a^2$,	$-8\lambda-12\mu$ $+(8\lambda+8\mu)\,Pa^2$,	$-(8\lambda+4\mu)\,Q\gamma$,	$-4\lambda\gamma-4\mu\gamma$ $+(8\lambda+4\mu)\,P\gamma a^2$
$(2\rho\omega p-\rho\omega^2)\,\gamma$,	$4\mu P\gamma a^2$,	0,	$-2\mu\gamma-(4\lambda+5\mu)\,P\gamma a^2$,	0,	$\rho p^2-\mu\gamma^2-(8\lambda+8\mu)\,Q$ $-(2\lambda+\mu)\,P\gamma^2a^2$,	8μ $+(16\lambda+8\mu)\,Pa^2$

in the opposite direction along the axis of z, we have standing waves of similar properties.

4. The symmetrical vibrations ($n = 0$) are of less physical interest, and are only mentioned in order to correct the erroneous results relating to them in my former paper.

The relative vibrations fall into two classes:—

(1) An approximately pure torsional vibration for which $\rho p^2 = \mu\gamma^2$.

(2) A vibration for which $\rho p^2 = E\gamma^2$, which is not, however, purely longitudinal in general. Elementary considerations, of moment of momentum, also suffice to establish this latter point.

ON CANONICAL FORMS

By E. K. Wakeford.

[Read February 12th, 1920.]

[This paper consists of the second part of a dissertation "General Theorems and Canonical Forms," written by Mr. Wakeford in the spring of 1916. As it was almost complete in itself it has been thought best to print it separately, with a few words of explanation to enable the reader to grasp the principle on which the argument rests.

Suppose it is required to show that a general ternary cubic in x, y, z can, by a linear transformation, be reduced to the form

$$X^3+Y^3+Z^3+6mXYZ;$$

then we have to *identify* the given cubic with

$$(\alpha_1x+\alpha_2y+\alpha_3z)^3+(\beta_1x+\beta_2y+\beta_3z)^3+(\gamma_1x+\gamma_2y+\gamma_3z)^3$$
$$+6m(\alpha_1x+\alpha_2y+\alpha_3z)(\beta_1x+\beta_2y+\beta_3z)(\gamma_1x+\gamma_2y+\gamma_3z),$$

i.e. we have to show that the last expression is capable of representing the general cubic in x, y, z.

Writing it as $\Sigma a_{pqr}x^py^qz^r$,

it suffices to prove that the a's are independent functions of the ten variables consisting of the α's, β's, γ's and m.

This amounts to proving that the Jacobian does not vanish identically, but it will be seen (§ 2) that Mr. Wakeford does not use the Jacobian explicitly. He rather assumes the existence of a relation between the a's, and makes a deduction therefrom which is easily turned into a test to determine whether a proposed canonical form is possible or not.*

* The idea of using the Jacobian seems to be due to Kronecker: cf. Lasker, *Math. Annalen*, Vol. 58 (1904), pp. 434–446. The methods there used are the same as Mr. Wakeford's in principle, but the form is not so convenient. Several references being made to Elliott (*Algebra of Quantics*) and Richmond ("On Canonical Forms," *Quarterly Journal of Math.*, Vol. 33 (1902), pp. 331–340), they are quoted as "Quantics" and "Richmond" respectively.

The writer's thesis is to establish the possibility or otherwise of a given reduction not to find the reducing process or the number of solutions.—J. H. G.]

1. The problem of reducing a q-ary p-ic, that is to say a quantic of order p in q variables, to a canonical form, consists of expressing the quantic

$$\sum_{s=1}^{m} a_s x_1^{{}_sp_1} x_2^{{}_sp_2} \dots x_q^{{}_sp_q}$$

in the proposed form

$$F \equiv \sum_{s=1}^{m} x_1^{{}_sp_1} x_2^{{}_sp_2} \dots x_q^{{}_sp_q} f_s(l_1, l_2, \dots, l_n),$$

where $l_1, l_2, \dots, l_n$ are (usually) independent variables, and

$${}_sp_1 + {}_sp_2 + \dots + {}_sp_q = p \quad (s = 1, 2, \dots, m).$$

It will be noticed that F appears to differ from some proposed forms in that no indefinite linear forms occur in it. If such a form X should occur, write

$$X \equiv x_1\lambda_1 + x_2\lambda_2 + \dots + x_q\lambda_q,$$

and the coefficients λ take their place among the variables l. Thus any number of indefinite quantics of linear or higher order may be disposed of, and F obtained in the form above.

2. The proposed form F is or is not canonical according as the functions $f_1, f_2, \dots, f_m$ are or are not independent.

Suppose that the form is not canonical, so that a relation $\psi(f) = 0$ exists. Then

$$\frac{\partial f_1}{\partial l_r}\frac{\partial \psi}{\partial f_1} + \frac{\partial f_2}{\partial l_r}\frac{\partial \psi}{\partial f_2} + \dots + \frac{\partial f_m}{\partial l_r}\frac{\partial \psi}{\partial f_m} = 0 \quad (r = 1, 2, \dots, n). \qquad (1)$$

Now

$$\frac{\partial F}{\partial l_r} \equiv \sum_{s=1}^{m} \frac{\partial f_2}{\partial l_r} x_1^{{}_sp_1} x_2^{{}_sp_2} \dots x_q^{{}_sp_q}.$$

The contragredient quantic Φ defined by

$$\Phi \equiv \sum_{s=1}^{m} \frac{\partial \psi}{\partial f_1} u_1^{{}_sp} u_2^{{}_sp_2} \dots u_q^{{}_sp_q} \frac{p!}{{}_sp_1!\ {}_sp_2! \dots {}_sp_q!}$$

where $u_1, u_2, \ldots, u_q$ correspond with $x_1, x_2, \ldots, x_q$ respectively, is apolar to $\partial F/\partial l_r$, in virtue of relation (1). Hence if F is not canonical the quantics $\partial F/\partial l_r$ are all apolar to a certain contragredient quantic. Note that this apolar form must exist for all values of l. [If $\partial\psi/\partial f_s$ $(s = 1, 2, \ldots, m)$ vanishes for a certain set of values of $l_1, \ldots, l_n$, *two* apolar forms instead of one are obtained.] In order therefore to prove that F is canonical, it is only necessary to find a particular set of values of $l_1, l_2, \ldots, l_n$, so that the quantics $\partial F/\partial l_r$ have no form apolar to them.

Again, if such an apolar form Φ exists for general values of l, the relations following hold good for suitable values of λ,

$$\frac{\partial f_1}{\partial l_1}\lambda_1 + \frac{\partial f_2}{\partial l_1}\lambda_2 + \ldots + \frac{\partial f_m}{\partial l_1}\lambda_m = 0,$$

$$\frac{\partial f_1}{\partial l_2}\lambda_1 + \frac{\partial f_2}{\partial l_2}\lambda_2 + \ldots + \frac{\partial f_m}{\partial l_2}\lambda_m = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\partial f_1}{\partial l_n}\lambda_1 + \frac{\partial f_2}{\partial l_n}\lambda_2 + \ldots + \frac{\partial f_m}{\partial l_n}\lambda_m = 0.$$

The Jacobian of the m functions with respect to any m of the n variables l must therefore vanish in general, and so vanish always. Hence the functions are not independent, and the form is not canonical. If there is no apolar form, then

$$\lambda_1 \frac{\partial F}{\partial l_1} + \lambda_2 \frac{\partial F}{\partial l_2} + \ldots + \lambda_n \frac{\partial F}{\partial l_n}$$

can be made to represent any q-ary p-ic for suitable values of $\lambda_1, \lambda_2, \ldots, \lambda_n$, and conversely.

3. It has hitherto been required that all implicit parameters in F, such as those contained in a linear form X, should be written out explicitly: *e.g.*

$$X \equiv \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_q x_q.$$

This may be avoided, for if

$$\frac{\partial F}{\partial X_r} \equiv x_r \frac{\partial F}{\partial X} \quad (r = 1, 2, \ldots, q)$$

is apolar to Φ, the quantic $\partial F/\partial X$ of order $p-1$ is apolar to Φ: and conversely if $\partial F/\partial X$ is apolar to Φ, then $x_r \cdot \partial F/\partial X$ is also apolar to Φ.

Thus suppose the quantic is the binary quartic, and

$$F \equiv X^4+6mX^2Y^2+Y^4.$$

Then

$$\tfrac{1}{4}\frac{\partial F}{\partial X} \equiv X^3+3mXY^2,$$

$$\tfrac{1}{4}\frac{\partial F}{\partial Y} \equiv 3mX^2Y+Y^3,$$

and

$$\tfrac{1}{6}\frac{\partial F}{\partial m} \equiv X^2Y^2,$$

are apolar to Φ. Put $m = 0$, and it is clear that Φ does not exist, and the form is canonical, since no binary quartic has X^3, Y^3, and X^2Y^2 apolar to it.

A similar method may be adopted in case an indefinite quantic of the second or higher order occurs in F. Thus in order to justify

$$F \equiv c_2c_3-c_1^2,$$

for the ternary quartic, where c_1, c_2, c_3 are conics, it is only necessary to consider that

$$-\tfrac{1}{2}\frac{\partial F}{\partial c_1} \equiv c_1,$$

$$\frac{\partial F}{\partial c_2} \equiv c_3,$$

$$\frac{\partial F}{\partial c_3} \equiv c_2;$$

and there is not always a class quartic to which those conics are apolar, as may be seen by considering the case

$$c_1 \equiv x_1^2, \quad c_2 \equiv x_2^2, \quad c_3 \equiv x_3^2.$$

Now take the case when the proposed form is the sum of n perfect p-th powers of linear form, that is to say

$$F \equiv X_1^p+X_2^p+\ldots+X_n^p.$$

If Φ exists X_r^{p-1} $(r = 1, 2, \ldots, n)$ is apolar to it. It will now be convenient to reciprocate the problem. Suppose S is a quantic of order p in

$x_1, x_2, \ldots, x_q$, and U^{p-1} is apolar to S. Considering S as denoting a hyper-surface, and U a point, it follows that the $(p-1)$-th polar of the hyper-surface with respect to the point vanishes, so that the point is a double point on the hyper-surface. Hence, if in the original problem Φ exists, there must be a hyper-surface of order p having n arbitrary double points. If for one set of points there is no such hyper-surface, F is canonical. If, on the other hand, a method of finding such a surface for the general case can be given, F is not canonical. A number of well known results follow immediately from this geometrical aspect of the problem, which often enables a solution to be obtained without putting pen to paper.

(1) The binary $(2n-1)$-ic can be expressed as the sum of n form X_r^{2n-1}, for no $(2n-1)$-ic can have double points at n different points.

(2) The ternary quadratic cannot be expressed as the sum of two squares, for the square of the line joining two points is a conic with double points at both of them.

(3) The ternary quartic cannot be expressed as the sum of five fourth powers, for the square of the conic through five points is a quartic with double points at all of them. [*Quantics*, § 230; *Richmond*, § 9.]

(4) The ternary quintic can be expressed as the sum of seven fifth powers, for if no six of seven points lie on a conic, it is easy to show that no quintic can have nodes at all of them. [*Richmond*, § 10.]

(5) The quaternary cubic can be expressed as the sum of five cubes, for no cubic surface can have five double points of which four do not lie on a plane. [Consider twisted cubics through the five points; *Richmond*, § 11.]

(6) The quinary cubic cannot be expressed as the sum of seven cubes, for through seven points in four dimensions passes a quartic curve, the chords of which generate a cubic hyper-surface containing the curve as a double curve. [*Richmond*, § 12.]

4. *The binary* $2n$*-ic*,

$$F \equiv X_1^{2n} + X_2^{2n} + \ldots + X_n^{2n} + \frac{2n!}{n!} m X_1^2 X_2^2 \ldots X_n^2.$$

In this case Φ, if it exists, has

$$\frac{1}{2n}\frac{\partial F}{\partial X_r} \equiv X_1^{2m-1} + \frac{(2n-1)!}{2^{n-1}} m X_1 \dots X_{r-1}^2 X_r X_{r+1}^2 \dots X_n^2,$$

and

$$\frac{\partial F}{\partial m} \equiv \frac{2n!}{2^n} X_1^2 X_2^2 \dots X_n^2,$$

apolar to it. Put $m = 0$. Then Φ is a $2n$-ic with $X_1, X_2, \dots, X_n$ as double points, that is Φ is $X_1^2 X_2^2 \dots X_n^2$. But this form is not always apolar to itself, unless $n = 1$; hence the proposed form F is canonical for all binary $2n$-ics except the quadratic.

The number of ways in which this reduction can be performed is interesting. For $n = 2, 3, 4$, the numbers are 3, 8, 5 respectively. [*Quantics*, § 211; Wakeford, "A Canonical Form of the Binary Sextic," *Messenger of Mathematics*, Vol. 43 (1914), pp. 25–28; and *Quantics*, § 227.]

It is clear that instead of $X_1^2 X_2^2 \dots X_n^2$, any function of $X_1, X_2, \dots, X_n$ which is not always apolar to $X_1^2 X_2^2 \dots X_n^2$ may be taken. [*Quantics*, § 225.]

5. Any quantic of order $p > 1$ may have its terms of the form $x_1^{p-1} x_2$ removed. Let F be the general p-ic in q variables $X_1, X_2, \dots, X_q$, without such terms as $X_1^{p-1} X_2$.

Consider any term of F, *e.g.* $k X_1^{p-2} X_2 X_3$. Φ if it exists will be apolar to $\partial F/\partial k$, *i.e.* $X_1^{p-2} X_2 X_3$. Now Φ may be written in terms of $U_1, U_2, \dots, U_q$, where U_1 is the common point of the linear forms $X_2, X_3, \dots, X_q$. Written thus, Φ evidently does not contain the term $U_1^{p-2} U_2 U_3$. Similarly for all the other terms of F. Hence Φ consists entirely of such terms as $U_1^{p-1} U_2$.

Consider $\partial F/\partial X_1$, which is apolar to Φ. Put the coefficients of all the terms of F zero, except the terms X_r^p. Then $\partial F/\partial X_1$ is $k X_1^{p-1}$. So Φ can contain no coefficient of U_1^{p-1}, or similarly of U_r^{p-1}. Hence Φ does not exist, since it has been shown to contain no terms except those of the form $U_1^{p-1} U_2$. Hence the proposed form is canonical. Particular cases are canonical forms of the binary cubic, quartic, and quintic, and of the ternary cubic.

A similar proof shows that all terms of the form $x_1^{p-r} x_2^r$ can be removed, where r is any fixed number. The following question now arises:—*Is*

there any ready test by which, given $q(q-2)$ terms of a quantic, it can be decided whether they are removable or not?

It is easy to prove by the method above that the following test is *necessary*:—*It must be possible to associate with each of the terms to be removed a different one of the ratios*

$$x_r/x_s \quad (r = 1, 2, \ldots, q\,;\ s = 1, 2, \ldots, q\,;\ r \neq s),$$

so that the product in each case is a term which is not to be removed.

I cannot prove this condition sufficient, though it seems to be so.

The following particular form is certainly sufficient:—*Choose any set of terms which are prime to one another, that is to say such that the same x_r cannot occur in two of them. These terms may be "isolated" (except in the case of linear forms), i.e. all terms which can be found by multiplying them by x_r/x_s may be removed.*

For instance, $x_1^p, x_2^p, \ldots, x_n^p$ are isolated if the terms $x_r^{p-1}x_s$ are removed.

6. *The lines on a cubic surface.*—It may be inadvisable to make explicit use of the apolar form Φ. It order to dispense with it, note that since the general q-ary p-ic contains m terms, and the forms $\partial F/\partial l_r$ are all apolar to Φ, a syzygy must exist between any m of the n forms $\partial F/\partial l_r$. Conversely, if that is true, Φ must exist. The simplest case is where $n = m$, then the form is canonical or not according as there does or does not exist a syzygy between the forms $\partial F/\partial l_r$. In dealing with a linear form X, the term in the syzygy corresponding to $\partial F/\partial X$ is $X' . \partial F/\partial X$, where X' is an arbitrary linear form. The following example is written out at length.

The study of the lines on a cubic surface may be started by taking

$$F \equiv X_1X_2X_3 - X_4X_5X_6$$

as the canonical form of a cubic surface. In order to justify this, write

$$X_r \equiv l_rx_1 + m_rx_2 + n_rx_3 + p_rx_4 \quad (r = 1, 2, \ldots, 6\,;\ p_r \neq 0).$$

Consider the twenty cubic surfaces

$$X_1X_2X_3, \quad X_2X_3x_1, \quad X_2X_3x_2, \quad X_2X_3x_3, \quad X_3X_1x_1, \quad X_3X_1x_2, \quad X_3X_1x_3,$$

$$X_1X_2x_1, \quad X_1X_2x_2, \quad X_1X_2x_3, \quad X_4X_5X_6, \quad X_5X_6x_4, \quad X_5X_6x_5, \quad X_5X_6x_6,$$

$$X_6X_4x_1, \quad X_6X_4x_2, \quad X_6X_4x_3, \quad X_4X_5x_1, \quad X_4X_5x_2, \quad X_4X_5x_3.$$

These surfaces are all apolar to Φ, if it exists. Hence, since there are only twenty terms in the equation of a cubic surface, a syzygy must connect the surfaces, viz.

$$X_2X_3X_1'+X_3X_1X_2'+X_1X_2X_3' \equiv X_5X_6X_4'+X_6X_4X_5'+X_4X_4X_6',$$

where neither side can vanish, since $p_r \neq 0$.

Consider the cubic surface represented by both sides of this syzygy. It contains the six lines in which X_2X_3, X_3X_1, X_1X_2, X_5X_6, X_6X_4, X_4X_5 respectively intersect. The plane X_1 meets the surface in two lines, viz. X_1X_2, X_1X_3. The three lines X_5X_6, X_6X_4, X_4X_5 accordingly meet X_1 in collinear points (or else on one of these two lines), and this is not the case for all sets of six planes X_r. Hence the syzygy is impossible and the form canonical.

INDEX.

INDEX OF PROCEEDINGS.

[*The numbers refer to pages.*]

Annual General Meeting, iv.

Auditor's Report, vi.

Council and Officers for the Session 1918–1919, iv.

Elections of Members, iv, vi, viii, xii, xvii, xxi, xxvii.

Exchanges of Publications, xxxi, xxxii.

Number of Members, xii.

Presents to Library, xxxi.

Treasurer's Report, iv.

INDEX OF PAPERS READ AND COMMUNICATIONS MADE TO THE SOCIETY DURING THE SESSION NOVEMBER, 1918—JUNE, 1919.

[*The names of authors are given and the numbers of the pages on which the titles of the papers, etc., will be found.*]

Ananda Rau, K., xxi, xxii.
Broderick, T. A., v.
Bromwich, T. J. I'A., ix.
Carslaw, H. S., xiii.
Cunningham, Lt.-Col. A., xxi, xxv.
Fréchet, M., vii, ix.
Hammond, J., xvii, xviii.
Hanumanta Rao, C. V., xxviii, xxix.
Hardy, G. H., vi, vii, xxi, xxii.
Hill, M. J. M., iv.
Hilton, H., xxi.
Hodgkinson, J., v.
Lewis, T. C., xii, xiii.
Littlewood, J. E., vi, vii, xxi, xxii.
Macdonald, H. M., iv.
MacMahon, Major P. A., xvii, xviii.
Miller, G. A., xxviii.
Milne, W. P., v, xxviii, xxix.
Mordell, L. J., iv, v, viii, xviii.
Pidduck, F. B., xxviii, xxx.
Priestley, H. J., v.
Ramanujan, S., xvii, xix, xx.
Shah, N. M., vi, viii.
Taylor, D. G., xxviii, xxix.
Watson, G. N., xxvii.
Wilson, B. M., vi, viii.
Young, W. H., xxi, xxiii, xxvii.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages; Roman numerals refer to Abstracts.*]

Bateman, H. The Nature of a Moving Electric Charge and its Lines of Electric Force, 95.

Carslaw, H. S. Diffraction of Waves by a Wedge of any Angle, 291.

Hardy, G. H. Additional Note on Two Problems in the Analytic Theory of Numbers, 201.

Hardy, G. H., and Littlewood, J. E. Abel's Theorem and its Converse, 205.

Hippisley, Col. R. L. A New Method of Describing a Three-Bar Curve, 136.

Hobson, E. W. On Hellinger's Integrals, 249.

Hodgkinson, J. A Detail in Conformal Representation, 268.

Ince, E. L. On the Continued Fractions connected with the Hypergeometric Equation, 236.

Milne, W. P. Determinantal Systems of Co-Apolar Triads on a Cubic Curve, 274.

——— A simple Condition for Co-Apolar Triangles, 336.

Milne, W. P., and Taylor, D. G. The Significance of Apolar Triangles in Elliptic Function Theory, 375, xxix.

Pidduck, F. B. The Vibrations and Stability of a Rotating Cylinder, 393, xxx.

Priestley, H. J. Note on the Values of n which make $\frac{d}{dx}\{P_n^{-m}(x)\}$ vanish at $x = a$, 266.

Proudman, J. On the Dynamical Equations of the Tides—Part I, Tidal Systems with Finite Freedom, 1.

——— On the Dynamical Equations of the Tides—Part II, Flat Seas, 21.

——— On the Dynamical Equations of the Tides—Part III, Oceans on a Sphere, 51.

Sommerville, D. M. Y. The Singularities of the Algebraic Trochoids, 385.

Turnbull, H. W. The Simultaneous System of Two Quadratic Quaternary Forms, 69.

Wakeford, E. K. On Canonical Forms, 403.

Young, A. The Electromagnetic Properties of Coils of certain Forms, 280.

Young, W. H. On the Connexion between Legendre Series and Fourier Series, 141.

——— On Series of Bessel Functions, 163.

——— On Non-Harmonic Fourier Series, 307.

——— On a Formula for an Area, 339, xxiii.

INDEX OF ABSTRACTS OF PAPERS NOT PRINTED IN FULL IN THIS VOLUME, ARRANGED ACCORDING TO THE NAMES OF AUTHORS.*

Ananda Rau, K. On Lambert's Series, xxi.

Bromwich, T. J. I'A. Some Operational Solutions in Conduction of Heat, ix.

Cunningham, Lieut -Col. A. Expression of Primes $p = 10w \pm 1$ in various Forms, xxv.

Hammond, J. On the Solution of the Quintic, xviii.

Hanumanta Rao, C. V. Some Considerations on the General Theory of Ruled Surfaces, xxix.

Hardy, G. H., and Littlewood, J. E. Applications of the Method of Farey Dissection in the Analytic Theory of Numbers, vii.

———— A Tauberian Theorem for Lambert's Series, xxii.

Lewis, T. C. General Pentaspherical Coordinates, xiii.

MacMahon, Major P. A. Divisors of Numbers and their Continuations in the Theory of Partitions, xviii.

Miller, G. A. Groups involving Three and only Three Operators which are Square, xxviii.

Mordell, L. J. A Statement by Fermat, v.

Ramanujan, S. Congruence Properties of Partitions, xix.

———— Algebraic Relations between certain Infinite Products, xx.

———— On the Relation between the Convergence of a Series and its Summability by Cesàro's Means, xxii.

Watson, G. N. The Zeros of Lommel's Polynomials, xxvii.

Wilson, B. M., and Shah, N. M. Numerical Data connected with Goldbach's Theorem, viii.

---

* References to Abstracts of papers printed in full in this volume will be found in the Index of Papers (p. 413).

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Bessel Functions: Young, W. H., 163; Watson, G. N., xxvii.
Biography (Obituary Notices), xxxiii.
Canonical Forms: Wakeford, E. K., 403.
Coils, Magnetic: Young, A., 280.
Combinatory Analysis: MacMahon, Major P. A., xviii; Ramanujan, S., xix.
Conformal Representation: Hodgkinson, J., 268.
Continued Fractions: Ince, E. L., 236.
Definite Integrals: Hobson, E. W., 249; Young, W. H., 339, xxiii.
Differential Equations: Ince, E. L. 236; Bromwich, T. J. I'A., ix.
Dynamics: Pidduck, F. B., 393, xxx.
Elasticity, Theory of: Pidduck, F. B., 393, xxx.
Electromagnetism: Bateman, H., 95; Carslaw, H. S., 291; Young, A., 280.
Elliptic Functions: Milne, W. P., and Taylor, D. G., 375, xxxix.
Equations, Theory of: Hammond, J., xviii; Watson, G. N., xxvii.
Fourier Series: Young, W. H., 141, 163, 307.
Functions of a Complex Variable: Hodgkinson, J., 268; Ince, E. L., 236.
Functions of a Real Variable: Hardy, G. H., and Littlewood, J. E., 205, xxii; Hobson, E. W., 249; Young, W. H., 141, 163, 307.
Geometry: Hanumanta Rao, C. V., xxix; Hippisley, Col. R. L., 136; Lewis, T. C., xiii; Milne, W. P., 274, 336; Milne, W. P., and Taylor, D. G., 375, xxxix; Sommerville, D. M. Y., 385; Young, W. H., 339, xxiii.
Groups, Theory of: Miller, G. A., xxviii.
Harmonic Functions: Priestley, H. J., 266; Young, W. H., 141, 163, 307.
Heat, Conduction of: Bromwich, T. J. I'A., ix.
Hydrodynamics: Proudman, J., 1, 21, 51.
Hypergeometric Series: Ince, E. L., 236.
Integration, Theory of: Hobson, E. W., 249.
Invariants: Turnbull, H. W., 69.
Kinematics: Hippisley, Col. R. L., 136.
Legendre Functions: Priestley, H. J., 266; Young, W. H., 141.
Numbers, Theory of: (Elementary) Cunningham, Lieut.-Col. A., xxv; Mordell, L J., v; (Analytic) Hardy, G. H., 201; Hardy, G. H., and Littlewood, J. E., vii; MacMahon, Major P. A., xviii; Ramanujan, S., xix, xx; Wilson, B. M., and Shah, N. M., viii.
Optics, Physical: Carslaw, H. S., 291.
Oscillations: Pidduck, F. B., 393, xxx.
Quadratic Forms: Turnbull, H. W., 69.
Series, Theory of: Ananda Rau, K., xxi, xxii; Hardy, G. H., and Littlewood, J. E., 205, xxxii; Ramanujan, S., xx.
Tides, Theory of: Proudman, J., 1, 21, 51.

Printed by C. F. Hodgson & Son, Newton Street, Kingsway, W.C. 2.

BIBL.
LONDIN.
UNIV.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 18.—PART 1.

CONTENTS.

ON THE DYNAMICAL EQUATIONS OF THE TIDES—PART I, TIDAL SYSTEMS WITH FINITE FREEDOM: DR. J. PROUDMAN 1

ON THE DYNAMICAL EQUATIONS OF THE TIDES—PART II, FLAT SEAS: DR. J. PROUDMAN 21

ON THE DYNAMICAL EQUATIONS OF THE TIDES—PART III, OCEANS ON A SPHERE: DR. J. PROUDMAN 51

THE SIMULTANEOUS SYSTEM OF TWO QUADRATIC QUATERNARY FORMS: MR. H. W. TURNBULL 69

Issued May 14th, 1919.

LONDON

PRINTED BY C. F. HODGSON & SON,

NEWTON STREET, KINGSWAY, W.C. 2;

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C. 4.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of fifteen (or in certain cases ten) guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the first Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County, Westminster and Parr's Bank, 1 St. James's Square, London, S.W.1, or to the Treasurer, Dr. A. E. Western, 35 Essex Street, Strand, London, W.C.2.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, which will be held in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The dates of meetings for the Session 1918–1919 are: Nov. 14, Dec. 12, Jan. 16, Feb. 13, March 13, April 24, May 15, and June 12. Cards giving the Programmes of the meetings as they occur can be had by Members a few days beforehand on application to C. F. Hodgson & Son, 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short Abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained. Abstracts of papers submitted for publication in full should be extremely brief; as a rule, such an Abstract should not exceed half a page of the *Proceedings*. When an Abstract refers to a communication which is not to be published in full, more space will be allowed for the Abstract, with a maximum of about two pages.

Communications for the Secretaries may be forwarded to them at the following addresses:—

1 Selwyn Gardens, Cambridge—T. J. I'A. Bromwich, Sc.D., F.R.S.

Trinity College, Cambridge—G. H. Hardy, M.A., F.R.S.

London Mathematical Society, Burlington House, W.1.

AUTHORS of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. *d.*	*s.* *d.*	*s.* *d.*
4 pages	Free	4 6	6 6	8 0
8 ,, ...	,,	6 0	8 6	10 6
12 ,, ...	,,	9 6	14 0	17 0
16 ,, ...	,,	10 6	16 0	20 0
32 ,, ...	,,	18 6	28 0	34 6

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. FRANCIS HODGSON, 89 Farringdon Street, E.C.4, at the following rates: –

Vol.	*I.	Nos.	1–8	...	Jan. 1865	to Nov. 1866	...	10*s.*
,,	*II.	,,	9–20	...	Nov. 1866	,, ,, 1869	...	16*s.*
,,	III.	,,	21–40	...	,, 1869	,, ,, 1871	...	20*s.*
,,	IV.	,,	41–66	...	,, 1871	,, ,, 1873	...	31*s.* 6*d.*
,,	V.	,,	67–75	...	,, 1873	,, ,, 1874	...	15*s.*
,,	VI.	,,	76–88	...	,, 1874	,, ,, 1875	...	21*s.*
,,	VII.	,,	89–103	...	,, 1875	,, ,, 1876	...	21*s.*
,,	VIII.	,,	104–123	...	,, 1876	,, ,, 1877	...	21*s.*
,,	IX.	,,	124–140	...	,, 1877	,, ,, 1878	...	21*s.*
,,	X.	,,	141–155	...	,, 1878	,, ,, 1879	...	18*s.*
,,	XI.	,,	156–165	...	,, 1879	,, ,, 1880	...	12*s.* 6*d.*
,,	XII.	,,	166–179	...	,, 1880	,, ,, 1881	...	16*s.*
,,	XIII.	,,	180–194	...	,, 1881	,, ,, 1882	...	18*s.*
,,	XIV.	,,	195–214	...	,, 1882	,, ,, 1883	...	20*s.*
,,	XV.	,,	215–232	...	,, 1883	,, ,, 1884	...	20*s.*
,,	XVI.	,,	233–252	...	,, 1884	,, ,, 1885	...	20*s.*
,,	XVII.	,,	253–279	...	,, 1885	,, ,, 1886	...	25*s.*
,,	XVIII.	,,	280–304	...	,, 1886	,, ,, 1887	...	25*s.*
,,	XIX.	,,	305–342	...	,, 1887	,, ,, 1888	...	30*s.*
,,	XX.	,,	343–369	...	,, 1888	,, ,, 1889	...	25*s.*
,,	XXI.	,,	370–398	...	,, 1889	,, ,, 1890	...	25*s.*
,,	XXII.	,,	399–429	...	,, 1890	,, ,, 1891	...	27*s.* 6*d.*
,,	XXIII.	,,	430–449	...	,, 1891	,, ,, 1892	...	20*s.*
,,	XXIV.	,,	450–474	...	,, 1892	,, ,, 1893	...	22*s.*
,,	XXV.	,,	475–499	...	,, 1893	,, ,, 1894	...	22*s.*
,,	XXVI.	,,	500–534	...	,, 1894	,, ,, 1895	...	30*s.*
,,	XXVII.	,,	535–574	...	,, 1895	,, ,, 1896	...	35*s.*
,,	XXVIII.	,,	575–611	...	,, 1896	,, ,, 1897	...	32*s.* 6*d*
,,	XXIX.	,,	612–657	...	,, 1897	,, ,, 1898	Pt. I. 20*s.* Pt. II. 20*s.*	
,,	XXX.	,,	658–678	...	,, 1898	,, Mar. 1899	...	20*s.*
,,	XXXI.	,,	679–709	...	April 1899	,, Dec. 1899	...	20*s.*
,,	XXXII.	,,	710–737	...	Jan. 1900	,, June 1900	...	20*s.*
,,	XXXIII.	,,	738–763	...	Nov. 1900	,, Feb. 1901	...	20*s.*
,,	XXXIV.	,,	764–789	...	Mar. 1901	,, April 1902	...	20*s.*
,,	XXXV.	,,	790–819	...	May 1902	,, Jan. 1903	...	20*s.*

* It was ordered by the Council (June 8, 1911) that Vols. I and II of the first Series should not be sold except to purchasers of not less than twenty volumes of that Series.

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s.* each.

A complete Index of all the papers printed in Vols. I to XXX of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25*s.* each Volume (of about 400 pp.). The volumes may be subscribed for in advance by the public at the above price, and may be obtained for this subscription either in separate Parts immediately on publication or in bound Volumes. Cases for binding the Volumes can be procured from the Publisher, price 1*s.* 6*d.* each.

Vols. 1 to 16 of Series 2 (1904–1918) have been published.

It was ordered by the Council (Feb. 13, 1896), "That separate Parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete set of the Proceedings up to the last completed volume.*"

CAMBRIDGE UNIVERSITY PRESS

Lectures on the Principle of Symmetry and its Applications in all Natural Sciences. By F. M. JAEGER, Ph.D., Professor of Inorganic and Physical Chemistry in the University of Groningen, Holland. Royal 8vo. With 170 diagrams. 20s net.

The author's purpose is to draw the attention of students of mathematics and natural philosophy in general to a principle of which the significance in the morphological description of objects, as well as in the definition of chemical and physical phenomena, is gradually becoming more and more evident in every domain of research.

Matrices and Determinoids. By C. E. CULLIS, M.A., Ph.D., Hardinge Professor of Higher Mathematics in the University of Calcutta. Volume II. Royal 8vo. 42s net

CONTENTS: Compound Matrices; Relations between the Elements and Minor Determinants of a Matrix; Some Properties of Square Matrices; Ranks of Matrix Products and Matrix Factors; Equigradent Transformations of a Matrix whose Elements are Constants; Some Matrix Equations of the Second Degree; The Extravagances of Matrices and of Spacelets in Homogeneous Space; The Paratomy and Orthotomy of Two Matrices and of Two Spacelets of Homogeneous Space.

Volume I, previously published, 21s net.

A Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., F.R.S., Sadleirian Professor of Pure Mathematics, and Fellow of Christ's College, Cambridge. Fourth edition. Demy 8vo. 12s 6d net

"There was a time when Professor Hobson's volume was the only British textbook in which the higher portions of the subject were adequately treated. Even now it has no serious rival. . . . We believe that 'Hobson' will continue to be the favourite textbook for those who wish to carry the subject beyond a merely elementary course."—*The Mathematical Gazette* on the Third Edition

The Principles of Projective Geometry applied to the Straight Line and Conic. By J. L. S. HATTON, M.A., Principal of the East London College. Royal 8vo. 10s 6d net

"This is a thorough treatise on the point, line, and conic developed by means of the principles of projective and perspective geometry. . . . It is an original work worthy to be put on a level with Chasles's *Traité des Sections Coniques*, with which it may well be compared, with the advantage in Mr Hatton's favour that in the fifty years that have elapsed since Chasles's work appeared (1865), many new theorems and properties of conics have been discovered which are here given either in the text or amongst the examples, of which the author has given a very large number, with hints or solutions, always neat, and more or less full, as required. . . . Mr Hatton has been fortunate in meeting with a field of research in which he has so few predecessors, and his clearness of statement and good readable style cannot fail to popularize both the method employed and the subjects dealt with."—*The Mathematical Gazette*

The Mathematical Theory of Electricity and Magnetism. By J. H. JEANS, M.A., F.R.S., formerly Stokes Lecturer in Applied Mathematics in the University of Cambridge. Third edition. Large Royal 8vo. 15s net.

"It would not be easy to praise too highly the introduction to mathematical physics with which Professor Jeans has provided the English student in this volume. . . . Professor Jeans is to be congratulated on the production of a volume which will certainly become the standard textbook on its subject."—*The Morning Post*

Elements of the Mathematical Theory of Electricity and Magnetism. By Sir J. J. THOMSON, O.M., F.R.S. Fourth edition. Crown 8vo. 10s 6d net.

"It may confidently be said that there is no class of student, using the term in its widest sense, who will not appreciate the contents. The mathematician will rejoice in the directness and luminosity of the proofs."—*The Electrician*

The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S. Demy 4to. Volumes I–XIII. 31s 6d net each. Index to the whole. 14s net.

The Collected Mathematical Papers of James Joseph Sylvester, F.R.S., D.C.L., LL.D., Sc.D. Edited by H. F. BAKER, Sc.D., F.R.S. Vol. I. 1837–1853. Vol. II. 1854–1873. Vol. III. 1870–1883. Vol. IV. 1882–1897. Royal 8vo. Buckram. 21s net each.

Royal Society Catalogue of Scientific Papers. Fourth Series (1884–1900). Vol. XVI, I—MARBUT. Demy 4to. Cloth, £5 5s net. Half-morocco, £6 net.

Fetter Lane, London, E.C. 4: C. F. Clay, Manager